The University of Canterbury
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A Theoretical Constructivisation of Mathematical Economics

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Abstract

This thesis deals with some problems in mathematical economics, looked at constructively; that is, with intuitionistic logic. In particular, we look at the connection between approximate Pareto optima and approximate equilibria. We then examine the classically vacuous, but constructively nontrivial, problem of locating the exact point where a line segment crosses the boundary of a convex subset of $\mathbb{R}^N$. We also prove the pointwise continuity of an associated boundary crossing mapping.

Turning to a rather different aspect of the theory, we discuss Ekeland's Theorem giving approximate minima of certain functions, as well as some fundamental notions in related areas of optimisation. The thesis ends with a discussion of some problems associated with the possible constructivisation of McKenzie's proof of the existence of competitive equilibria.

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Chapter 1

Some General Aspects of Constructive Mathematics

1.1 A Brief History of Constructive Mathematics

Constructive mathematics is an activity arising from concerns about the foundations on which mathematics is built. Until the late nineteenth century, mathematicians freely used a logic—classical logic—that, especially in highly abstract situations, gave rise to theorems with little or no computational content.

Consider, for example, the situation with Hilbert's Basis Theorem in algebra. This theorem originated in the search for fundamental invariants; this search was intended to produce those invariants explicitly. However, by a masterly use of classical logic, Hilbert proved that those invariants had to exist *without actually showing how they could be constructed.* This gave rise to the famous comment of the invariant–theorist Paul Gordan:

> *"That is not mathematics; that is theology."*

Although Hilbert's proof was not the first to establish the existence of a mathematical object by proving that its non–existence is contradictory, it was the most significant and dramatic proof of its type to date. After that, such "existential" proofs became standard in advanced mathematics, with few mathematicians taking any note of the loss of computational information that such a proof–tactic entailed.

One strong objector to the free use of existential methods in mathematics was Kronecker, who even went so far as to believe that irrational numbers were not properly constructed objects: he commented to Lindemann (who had proved the transcendentality of π):

> *"Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers are non–existent?"*

But it was not until 1907 that a fully constructive approach to mathematics was seriously put forward. This was done by the Dutch mathematician Luitzen Egbertus Jan Brouwer (1881–1966) whose 1907 doctoral thesis "Over der Grondslagen der Wiskunde" (On the Foundations of Mathematics) presented his views about mathematics as a free creation of the human mind, in which constructibility was an essential feature. Brouwer's philosophy, which covered more than just the philosophy of mathematics, was known as *intuitionism.* According to Brouwer's most famous pupil, Arend Heyting (1898–1980),

> *"The intuitionist mathematician proposes to do mathematics as a natural function of his intellect, as a free, vital activity of thought. For him, mathematics is a production of the human mind. He uses language, both natural and formalised, only for communicating thoughts, i.e., to get others or himself to follow his own mathematical ideas. Such a linguistic accompaniment is not a representation of mathematics; still less is it mathematics itself.*
>
> *It would be most in keeping with the active attitude of the intuitionist to deal at once with the construction of mathematics. The most important building block of this construction is the concept of unity which is*

the architectonic principle on which the series of integers depends. The integers must be treated as units which differ from one another only by their place in this series."

Although, in a sense, intuitionism can be regarded as a return to the pre–Hilbert era in which direct, constructive proofs were the norm and existential ones relatively rare, it was sufficiently radical that it found little acceptance in the general mathematical community. At least in part this was due to Brouwer's introduction of certain principles ("continuity" and "bar induction") that seem reasonable under his philosophy but that lead to results incompatible with classical mathematics. Nevertheless, for a time the great mathematician Hermann Weyl (1885–1955) was converted to intuitionistic mathematics, and Hilbert himself initially held Brouwer in very high regard; but eventually Brouwer and Hilbert fell out spectacularly, and Hilbert used all his power and prestige in the mathematical community to argue against Brouwer's views. (See [46], [89] for more details about this "Grundlagenstreit".)

A second approach to constructive mathematics arose in the late 1940s, under the leadership of A.A. Markov (1903–1979) in the Soviet Union. The constructive mathematics developed by Markov was based on the formal notion of a recursive algorithm, and dealt with recursive objects (like real numbers) and concepts (such as continuity). The resulting mathematics developed over the next twenty–five years contained many insights and some fascinating examples that showed that certain classical results (such as the monotone convergence theorem) failed when given a recursive interpretation. But it is fair to say that the combined resources of the intuitionists and the members of Markov's school did not produce enough positive mathematics to stimulate the interest of the general mathematical community, let alone lead that community to adopt a constructive approach to its activities. The received wisdom was still that of Hilbert:

> *"Forbidding a mathematician to make use of the principle of excluded middle is like forbidding an astronomer his telescope or a boxer the use of his fists."*

A major break–through occurred in 1967 with the publication of the monograph *Foundations of Constructive Analysis* by Errett Bishop (1928–1983). In a three–year period of intense research, Bishop developed a wealth of mathematics constructively, without adopting either Brouwer's special intuitionistic principles or the recursive framework of Markov. No longer could ardent proponents of classical mathematics seriously endorse Hilbert's claim (above), since Bishop had clearly demonstrated that deep results in such abstract areas as measure theory, spectral theory, and Banach algebras could be obtained constructively. The secret of Bishop's success (apart from his innate genius, which had already been shown in his early work in classical functional analysis and several complex variable theory) was his replacement of classical logic by the *intuitionistic logic* that Heyting [53] had abstracted from Brouwer's intuitionistic mathematical practice. It turned out that the use of intuitionistic logic and the normal objects of mathematics (for example, real numbers rather than recursive real numbers) was enough to exclude existential arguments and to ensure that all proofs were fully constructive.

But there was a price to pay: in order to extract constructive information, one often has to work a lot harder than the classical mathematicians. This was noted by Eric Schechter in his article "Constructivism is difficult" [81]:

> *"Constructive mathematics, with its stricter notion of proof, proves fewer theorems than classical mathematics does. A mainstream mathematician who wishes to learn constructivism must go through his or her entire catalogue of theorems, reevaluating each one by new criteria."*

1.2 Basic Notions of Constructive Mathematics

For convenience we label the three main varieties of constructive mathematics as **INT** (intuitionism), **RUSS** (Markov's recursive constructive mathematics), and **BISH** (Bishop's mathematics with intuitionistic logic). The fundamental notion of constructive mathematics is that of **algorithm**. A classical mathematician frequently will try to prove the existence of an object x by proving the impossibility of its nonexistence; a constructive mathematician will try to produce an algorithm that constructs x.

Are any differences between the definition of the notion of algorithm in the three groups specified above? In **BISH** and **INT** an algorithm is defined as a step–by–step, deterministic computation operated by a human brain in a finite period of time. Some examples of algorithms in this context are the description of a recursive function, the deductive steps of a serious formula, and the inductive method of proving a mathematical statement. In **RUSS** the role of the human brain is taken by a (notional) computer that works with a clearly specified programming language. An algorithm is, essentially, a syntactically correct program in that language.

At least formally, **INT** and **RUSS** can each be regarded as **BISH** plus some extra principles. In the case of **INT**, those principles are the continuity and bar induction introduced by Brouwer. In **RUSS**, the principle is Church's Thesis—every algorithm is a recursive algorithm—and sometimes also Markov's Principle (which we shall deal with later). Thus **INT** and **RUSS** are models of **BISH**. It is important to realise that classical mathematics—**CLASS**—is also a model of **BISH**: every theorem proved in **BISH** is also, without change, a theorem of **CLASS**. Thus it may not be unreasonable to regard **BISH** as the constructive core of **INT**, **RUSS**, and **CLASS**.

The consistency of both **INT** and **RUSS** with **BISH** can be used to reveal the limitations of **BISH**. For example, it is provable in **INT** that every function from $[0,1]$ to the real line $\mathbb{R}$ is uniformly continuous; whereas there is an example in **RUSS** of a continuous function from $[0,1]$ to $\mathbb{R}$ that is not uniformly continuous. We conclude that it is impossible to prove, within **BISH**, either that every continuous function from $[0,1]$ to $\mathbb{R}$ is uniformly continuous or that that statement is false. For such reasons, in real analysis Bishop was led to consider only functions that were uniformly continuous on compact subsets of $\mathbb{R}$.

In order to understand **BISH**, we now need to explain informally the principles of intuitionistic logic. In the following, we let P, Q denote mathematical statements.

- To prove $P \wedge Q$, we produce a proof of P and a proof of Q.
- To prove $P \vee Q$, either we produce a proof of P or we produce a proof of Q. (It is not enough, constructively, to prove the impossibility of $\neg P \wedge \neg Q$.)
- To prove $P \Rightarrow Q$, we produce an algorithm which, applied to any proof of P, converts it to a proof of Q. (Note that we do not require P to have a proof here; our algorithm takes a proof of P, if we have one, and produces from it a proof of Q.)
- To prove $\neg P$, we prove $P \Rightarrow (0 = 1)$.
- To prove $\exists x \in A\ P(x)$, we require an algorithm that produces a certain object x, together with a proof that $(x \in A \Rightarrow Q)$.
- To prove $\forall x \in A\ P(x)$, we require an algorithm that, applied to an object x and a proof that $x \in A$, produces a proof that $P(x)$. (Note that the proof of $P(x)$ will require not just the data describing x, but also the data constituting a proof that $x \in A$.)

Given these proof–principles, can we provide a decent example of a non–constructive, but classically valid, proposition? We can. An important example is the **limited principle of omniscience (LPO)**:

> For each binary sequence $(a_n)_{n=1}^{\infty}$, either $a_n = 0$ for all n or else there exists n such that $a_n = 1$.

If this were provable using intuitionistic logic, then we would have an algorithm which, applied to any binary sequence $(a_n)_{n=1}^{\infty}$, would either establish that $a_n = 0$ for all n or else output a positive integer N such that $a_N = 1$. To see how unlikely such an algorithm is, for each positive integer n let $P(n)$ denote the statement

> There exists $k < n$ such that a block of 99 consecutive digits 9 occur from the kth digit in the decimal expansion of π.

Define $a_n = 0$ if $\neg P(n)$, and $a_n = 1$ if $P(n)$. Then $(a_n)_{n=1}^{\infty}$ is an increasing binary sequence. If LPO applied to it, we would have either a constructive proof that the decimal expansion of π contains no blocks of 99 successive digits equal to 9, or else an explicit instance of a positive integer K such that the block of 99 digits starting from the Kth in that expansion consists entirely of 9s. Given that we could replace 99 by 999, 9999, 999^{999}, ... in this argument, it seems extremely unlikely that such a proof will ever materialise. In turn, this leads us to disbelieve the constructive validity of LPO.

There is another argument in favour of rejecting LPO as a constructive principle: it is provable false in both **INT** and **RUSS**; see [38]. Thus we are led to regard LPO, and any proposition that implies it constructively, as essentially nonconstructive. This has implications for the constructive theory of the real line $\mathbb{R}$. For example, we cannot expect to prove constructively the statement

$$\forall x \in \mathbb{R}\ \ (x = 0 \vee x \neq 0) \tag{1.1}$$

(where $x \neq 0$ means $|x| > 0$). For if $(a_n)_{n=1}^{\infty}$ is any binary sequence, and we apply (1.1) to the real number

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n},$$

then either we have $x = 0$, in which case $a_n = 0$ for all n, or else we can produce N such that $x > 1/2^N$; in the latter case, we must have $a_n = 1$ for some $n \leq N$. We conclude that (1.1) implies LPO and is therefore essentially nonconstructive.

An example of the type just given, in which we show that a certain classical proposition implies LPO or some other essentially nonconstructive principle, is called a **Brouwerian example**.

Fortunately, the constructive theory of reals includes principles (either introduced axiomatically, as in [22], or proved as consequences of the definition of real numbers, as in [41] that enable us to circumvent the problems caused by the essentially nonconstructive nature of (1.1)). One such principle is

$$a < b \Rightarrow \forall x \in \mathbb{R}\ (a < x \vee x < b).$$

This allows us to split proof into overlapping cases of the form "a certain number is positive" and "a certain number is less than ε", where $\varepsilon > 0$.

The following are also essentially nonconstructive principles about binary sequences.

- The **lesser limited principle of omniscience (LLPO):** For each binary sequence $(a_n)_{n=1}^{\infty}$ such that $a_m a_n = 0$ whenever $m \neq n$, either $a_{2n} = 0$ for all n or else $a_{2n+1} = 0$ for all n. (The condition "$a_m a_n = 0$ whenever $m \neq n$" says that $(a_n)_{n=1}^{\infty}$ has at most one term equal to 1.)

- **Markov's Principle (MP):** For each binary sequence $(a_n)_{n=1}^{\infty}$,

$$\neg\forall n\,(a_n = 0) \Rightarrow \exists n\,(a_n = 1).$$

Of these, Markov's Principle is the most controversial, in that it is accepted, perhaps with reluctance, by some practitioners of **RUSS**. The underlying idea behind accepting MP is that if we can rule out the possibility that all terms a_n are 0, then by systematically inspecting $a_1, a_2, a_3, \ldots$, we are guaranteed to produce N with

$a_N = 1$. The trouble is that, in advance of our search, we have no idea of how long we have to search in order to find this N; in other words, the search embodied in MP is an unbounded one. For that reason, practitioners of **BISH** avoid MP. (Note that MP is inconsistent with Brouwer's theory of the creating subject [48], an extension of **INT**.)

The third chapter of this thesis deals with finding the point at which we cross the boundary of a convex set C as we move along a line segment from the interior of C to the exterior. We now give a Brouwerian example to show that there is a problem with such boundary crossings in the absence of convexity. We first note that the classical proposition

$$\forall x \in \mathbb{R} \ \ (x \geq 0 \vee x \leq 0)$$

implies LLPO: this is seen by considering the real number

$$\sum_{n=1}^{\infty} (-1)^n \frac{a_n}{2^n},$$

where $(a_n)_{n=1}^{\infty}$ is any binary sequence with at most one term equal to 1. Now consider any real number a with $|a| \ll 1$. Let S be the closed polygonal subset of $\mathbb{R}^2$ with vertices

$$(0,-1), \ \left(\frac{1}{3}, a\right), \ \left(\frac{2}{3}, a\right), \ (1,1), \ (1,-1), \ (-1,1).$$

Then $(0,0)$ is an interior point of S, and $(1,0)$ is bounded away from S. Where does the segment L joining these two points meet the boundary ∂S of S? Suppose it does so at the point (ξ, η). Then either $\frac{1}{3} < \xi$ or $\xi < \frac{2}{3}$. In the first case we cannot have $a > 0$, so $a \leq 0$; in the second case, we cannot have $a < 0$, so $a \geq 0$. Thus if we can find an exact boundary crossing point for L, then we can prove that $a \leq 0$ or $a \geq 0$. In other words, the statement,

> For each subset S of $\mathbb{R}^2$ and each line segment L that meets both the interior and the exterior of S, there exists a point at which L intersects ∂S,

implies LLPO and is therefore essentially nonconstructive.

It is worth noting that the set S in this example is compact—that is, complete and totally bounded. For it is closed in the complete space $\mathbb{R}^2$ and so is complete. To prove total boundedness, let $\varepsilon > 0$ and let T be the closed polygonal subset of $\mathbb{R}^2$ with vertices

$$(0,-1), \left(\frac{1}{3},\varepsilon\right), \left(\frac{2}{3},\varepsilon\right), (1,1), (1,-1), (-1,1).$$

It is not hard to construct an ε–approximation to T in the Euclidean metric on $\mathbb{R}^2$. Either $0 < |a|$ or $|a| < \varepsilon$. In the first case, as for T, so for S we can construct an ε–approximation. In the second case, it is easily shown that $T \subset S$ and that every point of S is at most 2ε from some point of T; so our ε–approximation to T is a 3ε–approximation to S. Hence S is totally bounded. Thus not even the addition of compactness to the requirements on our set will enable us to find exact boundary crossings.

The foregoing Brouwerian example also shows that the classical intermediate value theorem fails to hold constructively. However, for all practical purposes, the following two constructive versions of that theorem suffice.

- If $f : [0,1] \to \mathbb{R}$ is continuous and $f(0)f(1) < 0$, then for each $\varepsilon > 0$ there exists $x \in (0,1)$ such that $|f(x)| < \varepsilon$.

- If $f : [0,1] \to \mathbb{R}$ is continuous, $f(0)f(1) < 0$, and f is **locally nonzero** in the sense that for each $x \in [0,1]$ and each neighbourhood V of x there exists $y \in V$ with $f(y) \neq 0$, then there exists $x \in (0,1)$ such that $f(x) = 0$.

See [9] (Chapter 2, page 40, Theorem 4.8) for more on the intermediate value theorem.

So far, we have concentrated on essentially nonconstructive principles related to sequences of integers. These are special cases of more general, logical principles that are essentially nonconstructive, such as the **law of excluded middle (LEM),**

$$P \vee \neg P,$$

or, alternatively,

$$\neg\neg P \Rightarrow P,$$

and the **weak law of excluded middle,**

$$\neg P \vee \neg\neg P.$$

Occasionally we prove that a proposition is essentially nonconstructive by showing that it implies one of these stronger principles. What is the role of the axiom of choice in constructive mathematics? The full form of this axiom is known to imply the law of excluded middle [51] and so is of no use to us. However, most constructive mathematicians allow both the **principle of countable choice,**

> If for each positive integer n there exists $x \in A$ such that $P(n, x)$, then there is a mapping $f : \mathbf{N}^{+} \to A$ such that $P(n, f(n))$ for all n,

and the **principle of dependent choice,**

> If $a \in A$ and for each $x \in A$ there exists $y \in A$ such that $P(x, y)$, then there exists a sequence $(a_n)_{n=1}^{\infty}$ in A such that $a_1 = a$ and $P(a_n, a_{n+1})$ for each n.

However, some authors, notably Richman [78], are critical of the use of even these two choice principles in constructive mathematics.

1.3 Idealisation Much More Modest in Constructive Mathematics

To anyone already interested in the foundations of mathematics, the investigation here proposed will seem quite evidently purposeful. But what of the economists?

What possible advantage could accrue to economists if the present undertaking succeeds? If you are an economist, why study mathematical economics constructively, in the ways that this thesis explores?

There are several broad potential gains that we wish to detail. Naturally, economics, like any practical science, has a theoretical edifice that is the more perspicuous and the simpler to use the more carefully considered it is. Constructive proofs work with weaker logical assumptions than do classical proofs, and thus wherever they succeed they represent the attainment of a fuller, more meticulous, more exacting and precise understanding of a subject matter.

This can be advantageous in various indirect ways, for example heuristically. Because of the greater elegance, accuracy, specificity, and conciseness of constructive results, an investigator who is used to dealing in them is the more likely to achieve important new theoretical inspiration. Also, because the constructive version of some part of economic theory will generally be a modification or completion of an earlier, classical counterpart, if the classical theory was in any way ill worked out or erroneous, then the work towards a constructive version can detect these difficulties and correct them. Thus the hard work of producing results constructively can often alter a theory in ways that even from the classical vantage point constitute improvements.

In addition there are direct advantages. Constructive techniques carry the theoretician always only to where computers also can go. The sharper logical system means that results speak directly to the way in which the theory can be implemented by computer systems. A time factor in the application of theory is minimised. The minimisation of that time factor could, for an economist working in industry say, help assure a maximisation of profit. To the extent that constructive techniques carry theorists to new and improved forms of understanding, economists would be not only churlish but even downright foolhardy to resist the enterprise. Some crises in the international economy in recent decades have been blamed on faulty economic theory. Economists cannot afford to shun a new approach that promises to clarify and improve the ways in which they think.

Philosophically speaking, the edifice of extant economic theory looks like shaky knowledge. It trades in idealisations, and would not begin to achieve mathematical order and elegance without those. The edifice seems the shakier, however, the bolder or more immodest are the idealising assumptions that are made. Here, the constructive approach works a salutary effect. For example, economists often assume continuity. That is to say, across a wide spectrum of cases, they find that economic theory is better served by the ignoring of granularity. Economics makes better headway by treating its variables as continuous. However, in classical mathematics the assumption of continuity is something breathtakingly bold. When classical mathematicians introduce, say, a real–valued utility function in order to model certain economic phenomena, they commit themselves to the reality of an infinite order of potential utilities, the cardinality of which is strictly larger than that of the counting numbers. They presuppose moreover that that higher cardinality either determinately is or determinately is not the next higher cardinality after that of the counting numbers. But they find it impossible to discover any good grounds for saying which. That is to say, they cleave to a bivalent view of truth, despite many indicators that this assumption is baseless. Yet all these commitments are entirely without practical upshot within the sphere of applied economics. It seems better to cleave to more modest assumptions, and this the constructivists accomplish, simply by dint of their care about the foundations of mathematics.

1.4 Overview of the Thesis

After this first, introductory chapter, the thesis deals primarily with problems in, or arising from, the constructive theory of microeconomics. The main idea in the second chapter is the relation between notions of approximate equilibrium and approximate Pareto optimum, extending and clarifying the well–known classical equivalence between exact equilibria and Pareto optima [84] (Chapter 2). Our techniques of proof are based on classical counterpart but require much more careful estimation and attention to details that are often classically irrelevant (such as the locatedness of certain convex sets, which is needed before we can apply the separation theorem).

Chapter three deals with a geometrical problem arising from McKenzie's classical approach to proving the existence of economic equilibria. Work of Bridges *et al.* [40] has shown that under reasonable conditions on a subset S of a Banach space X, if we move along a line segment that starts at a point ξ in S° and ends outside S, then we get arbitrarily close to the boundary of S; in general, this is the best one we can do. However, we show in Chapter 3 that when $X = \mathbb{R}^N$ and S is located and convex, we can pinpoint the exact point at which the line segment from ξ crosses the boundary of S; moreover, the mapping that takes a point outside S to the corresponding boundary crossing point is continuous.

In the first part of the fourth chapter we prove a constructive version of Ekeland's theorem on the existence of approximate minima of a uniformly continuous mapping from a compact metric space into the set $\mathbb{R}^{0+}$ of nonnegative real numbers. We then investigate semicontinuity, and conditions which are equivalent to the (local) total boundedness of the strict lower sections of a function.

Chapter 5 covers a number of topics connected with McKenzie's proof. In particular, it deals with approximations to the polar and normalised polar of the production set of an economy, and points towards future research.

1.5 Notations

The following notations will be used in the Thesis:

$diam\,(F_n)$ denotes the diameter of the set F_n.

$\mathrm{dom}(f)$ denotes the domain of a function f.

$\mathbf{F}$ denotes either the real line $\mathbb{R}$ or the extended real line $\overline{\mathbb{R}}$.

H denotes a real (complex) Hilbert space.

$L(f)$ denotes the subgraph of a strongly extensional partial mapping f of a metric space X into $\mathbb{R}$ with nonempty domain.

Y denotes the aggregate production set.

Y_j denotes the production set.

X denotes the aggregate consumption set.

X_i denotes the consumption set.

$(\xi_i, \rightarrow)$ denotes the strict upper contour set.

$[\xi_i, \rightarrow)$ denotes the upper contour set.

$(\leftarrow, \xi_i)$ denotes the strict lower contour set.

$(\leftarrow, \xi_i]$ denotes the lower contour set.

$\sim S$ denotes the complement of a subset S of a metric space E.

$-S$ denotes the metric complement of a subset S of a metric space E.

$S^l(f, \lambda)$ denotes a lower section.

$S^{sl}(f, \lambda)$ denotes a strict lower section.

$S^{su}(f, \lambda)$ denotes a strict upper section.

$S^u(f, \lambda)$ denotes an upper section.

∂Y is the boundary of a set Y in a metric space.

Chapter 2

Approximate Equilibria and Pareto Optima

2.1 Informal Preliminaries

The Italian economist Vilfredo Pareto (1848–1923) proved that when a competitive market reaches the state of equilibrium, the outcome is Pareto–efficient (Pareto–optimal) [70]. Pareto actually himself coined the name "Pareto principle" which states that if the change is to be a Pareto improvement then everyone gains from it (the weaker version) or at least that if it is a Pareto improvement then some gain and nobody loses (the stronger version). Let $\omega S1$ and $\omega S2$ be two distinct economic situations or states. A different way of expressing the two forms of the Pareto principle is then that we should choose $\omega S2$ in preference to $\omega S1$ when everybody judges that they are better off in $\omega S2$ (the weaker version) or at least when both nobody judges that $\omega S1$ is better and at least one person judges that $\omega S2$ is better (the stronger version).

Pareto doubted whether it is possible to identify or rationally compare levels of aggregate interpersonal utility. Thus he did not accept the utilitarian view that a society is efficient when it maximises aggregate utility. He intended his notions to be a replacement for this view. In the discussion below we shall depart from Pareto to the extent that we find reason to introduce an understanding of utility that we shall show is well defined provided only that those other concepts are granted to us which are in any case core to Pareto's discussion. However the use that we make of this concept is directed precisely to a discussion of the Pareto principle.

Pareto assumed that all people are capable of choosing whether they are better off in one state, $\omega S1$ say, or another, $\omega S2$ say, and thereby concluded that if everyone preferred $\omega S2$ then social welfare must be greater in $\omega S2$, whereas if some preferred $\omega S1$ and some $\omega S2$, then we would be in no position to conclude which state is better. The point about competitive markets is that under ideal conditions they will make it salient for us to consider states between which Pareto comparisons of preferability can be made.

Many economists treat the Pareto principle as self–evident. Typically economists debate not whether the principle is true but rather whether it is all widely applicable. On the one hand one suspects that there are many states in the world, such as $\omega S1$ in which the water supply is fluoridated at some financial cost and $\omega S2$ in which the water supply is not fluoridated and that cost is saved, between which decisions have to be taken but neither of which is Pareto preferable to the other, since there are both winners and losers in both states. On the other hand one also suspects that even to the extent that Pareto preferability is an operative concept, there often will be many Pareto–efficient states to choose between.

Economists are apt to suppose that markets work precisely so as to make the concept of Pareto preferability applicable, and that they moreover reach equilibrium for just the reasons that Pareto points out precisely at some Pareto–efficient state. Thus if nothing socialist is done about the fluoridation of the water supply then toothpaste companies will compete with one another to provide that kind of protection through toothpaste. Consumers will choose to spend a little more on toothpaste but will benefit from stronger teeth. Toothpaste companies will profit, yet the consumers will also remark that they are better off. In short, in the equilibrium market conditions everyone will be better off.

Of course the optimality of the Pareto–efficient outcome here can be questioned. Society might have moved from the situation of no fluoride and poor teeth to one in which fluoride is dispensed through the water supply. The costs of such a measure would be borne by taxpayers and more highly taxed people might well be made on balance worse off. However, given that there are relatively few such people, the aggregate benefit might be far greater than the aggregate loss. When we move from $\omega S1$ to $\omega S2$ here many badly–off people gain but a few well–off people lose. We might think there is a gain in the overall efficiency. Clearly however we cannot make such a judgement using the original Pareto principle by itself.

We do not intend to explore the possible limits of the Pareto principle. Rather we shall explore its scope. We are interested to discuss the idealising assumptions on which the Pareto principle depends. Much idealisation is required even if one approaches the discussion constructively. However, as we intend to show, in some ways the constructivist's idealisations are more modest and in that way more acceptable than those that a classical mathematician must make.

2.2 Formal Preliminaries

In this chapter we introduce, and examine the relations between, natural notions of approximate chosen point, approximate Pareto optimum, and approximate equilibrium 1. For example, we show that, in senses made precise in Theorem 2.3.1 and Corollary 2.4.4, an approximate equilibrium 1 gives rise to an approximate Pareto optimum, which, in turn, consists of approximate chosen points for each of the consumers.

We assume that there are a finite number m of **consumers** and a finite number n of **producers**. Consumer i has **consumption set** $X_i \subset \mathbb{R}^N$, where a **consumption vector** $x_i = (x_{i_1}, \ \dots \ , x_{i_N}) \in X_i$ is interpreted as follows: x_{i_k} is the quantity of the kth commodity (a good or a service) taken by consumer i when he chooses the consumption vector x_i. Producer j has **production set** $Y_j \subset \mathbb{R}^N$, where the kth entry in the **production vector** $y_j = (y_{j_1}, \ \dots \ , y_{j_N}) \in Y_j$ is interpreted as the amount of the kth commodity produced by producer j under her adopted production schedule. Other important sets in this context are the **aggregate consumption set**

$$X = X_1 + \ \cdots \ + X_m$$

and the **aggregate production set**

$$Y = Y_1 + \ \cdots \ + Y_n.$$

We assume that consumer i has an initial endowment of commodities, represented by the vector $\bar{x}_i = (\bar{x}_{i_1}, \ \dots \ , \bar{x}_{i_N})$. The **total initial endowment** of all consumers is then

$$\bar{x} = \bar{x}_1 + \ \cdots \ + \bar{x}_m \in X.$$

We say that an element $(y_1, \ \dots \ , y_n)$ of $Y_1 \times \ \cdots \ \times Y_n$ is an **admissible array of production vectors**; and that an element $(x_1, \ \dots \ , x_m)$ of $X_1 \times \ \cdots \ \times X_m$ is a **feasible array of consumption vectors** if there exists an admissible array $(y_1, \ \dots \ , y_n)$ of production vectors such that

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j + \bar{x}.$$

Intuitively, a feasible array is one that can be obtained by a distribution of the total initial endowment and the total of the production vectors under some production schedule.

A **price vector** is simply a unit vector in $\mathbb{R}^N$; the kth component p_k of p is the price of one unit of the kth commodity. Thus the total cost to consumer i of the consumption vector x_i is $\langle p, x_i \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^N$; and the profit to producer j of the production vector y_j is $\langle p, y_j \rangle$.

We assume that the preferences of consumer i are represented by a binary relation $\succ_i$ of **strict preference** satisfying the following two conditions:

P1 $x \succ_i y \Rightarrow \neg (y \succ_i x)$;

P2 $x \succ_i y \Rightarrow \forall z \in X_i \ (x \succ_i z \vee z \succ_i y)$.

Clearly this involves significant idealisation. A person might have in general a preference, for, say, foreign travel over the latest in designer clothing. But she might find herself in a shoe shop with a very nice pair of shoes begging to be bought. She forgets herself and buys them. To this extent she shows that her preferences are volatile. We could take this sort of volatility into account by parameterising to time the relation of strict preference which we have just discussed. But during any short interval of time, only a tiny fraction of the dispositions of a person are at all engaged. So it could become very artificial to suppose that the preferences that she has that are ordered by $\succcurlyeq_i$ are at all thoroughgoing. Moreover if we consider a long enough time or enough people at once then the volatility of actual human behaviour may well appear so much at the level of minute detail that it matters not at all to us. In that case we would clearly compromise our theory the more by taking into account the fine–grained volatility than by choosing to ignore it. We bring to our theory not only greater mathematical elegance but also better applicability by insisting on an idealisation according to which people not only know themselves completely well but also are at all times maximally rational (i.e., prudent) in the light of that knowledge.

We define corresponding relations $\succcurlyeq_i$ of **preference–indifference**, and of **indifference**, as follows:

$$x \succcurlyeq_i y \quad \text{if and only if} \quad \forall z \in X_i \ (y \succ_i z \Rightarrow x \succ_i z), \text{ and}$$
$$x \sim_i y \quad \text{if and only if} \quad (x \succcurlyeq_i y \wedge y \succcurlyeq_i x).$$

The informal meaning of $x \succcurlyeq_i y$ is that consumer i finds x at least as attractive as y; $x \succ_i y$ means that he strictly prefers x to y; and $x \sim_i y$ signifies that he does not mind which of x or y he obtains.

Proposition 2.2.1 *The relations $\succ_i$, $\succcurlyeq_i$, and $\sim_i$ are transitive and have the following properties:*

(i) $x \succcurlyeq_i y$ *if and only if* $\neg\,(y \succ_i x)$.

(ii) *If either* $x \succ_i y$ *or* $x \sim_i y$, *then* $x \succcurlyeq_i y$.

(iii) *If either* $x \succ_i y \succcurlyeq_i z$ *or* $x \succcurlyeq_i y \succ_i z$, *then* $x \succ_i z$.

Proof. First assume that $x \succ_i y$ and $y \succ_i z$. Then, by P2, either $x \succ_i x$, which is ruled out by P1, or else, as must be the case, $x \succ_i z$. Next, assume that $x \succcurlyeq_i y$ and $y \succcurlyeq_i z$. For all ζ with $z \succ_i \zeta$ we have $y \succ_i \zeta$, since $y \succcurlyeq_i z$, and hence $x \succ_i \zeta$, since $x \succcurlyeq_i y$. Thus $x \succcurlyeq_i z$. The transitivity of $\sim_i$ is now immediate.

If $x \succcurlyeq_i y$ and $y \succ_i x$, then, by definition of $\succcurlyeq_i$, we have $x \succ_i x$, which is impossible in view of P1. This proves (i). To prove (ii), let $x \succ_i y$. Then for all z with $y \succ_i z$, we have $x \succ_i z$, by the transitivity of $\succ_i$. Hence $x \succcurlyeq_i y$. It is clear from the definition of $\sim_i$ that if $x \sim_i y$, then $x \succcurlyeq_i y$. This completes the proof of (ii). As for (iii), suppose, for example, that $x \succ_i y \succcurlyeq_i z$. By P2, either $x \succ_i z$ or $z \succ_i y$; since the latter is ruled out by (i), we have $x \succ_i z$. The other case is proved similarly.

q.e.d.

It is convenient to introduce consumer i's **upper contour set at** x,

$$[x, \rightarrow) = \{\xi \in X_i : \xi \succcurlyeq_i x\},$$

his **strict upper contour set at** x,

$$(x, \rightarrow) = \{\xi \in X_i : \xi \succ_i x\},$$

and his **strict lower contour set at** x,

$$(\leftarrow, x) = \{\xi \in X_i : x \succ_i \xi\}.$$

How should we interpret from an economic view–point the sets $[x, \rightarrow)$, $(x, \rightarrow)$, and $(\leftarrow, x)$ described mathematically above?

A consumer i desires, from a given shop, a dishwasher (denoted by d), a toaster (denoted by t), a radio (denoted by r), and a washing machine (denoted by wm), but not equally. About everything else in the shop she is uninterested. Among the four items she desires, her preference is, let us say, concentrated on the dishwasher. Next she would prefer the washing machine; her desire for it is not as great as that for the dishwasher because, let us say, of the local availability of a Laundromat. She desires the radio for entertainment, but it is not as important to her as respectively the dishwasher and the washing machine. Her desire for the toaster is also relatively modest and in fact she has no relative preference either way between a radio and a toaster. Under these circumstances the consumer i's contour set at r is

$$[r, \rightarrow) = \{d, t, wm\}.$$

The i's strict upper contour set at r is

$$(r, \rightarrow) = \{d, wm\}.$$

The i's strict lower contour set at r is

$$(\leftarrow, r) = \phi,$$

if we look at the set of preferences from the shop. If we regard as the consumption set all the products from the shop, the strict lower contour set at r is

$$(\leftarrow, r) = C - B,$$

where C represents the set of products from the shop, and B the set of products that the consumer i chooses to buy. An exception to this would be if consumer i has had sufficient cash to purchase three things only; the dishwasher, the washing machine, and one or the other but not both of the toaster and the radio between which she has no relative preference either way. Let us suppose that she has mentally flipped a coin between those and bought the radio. Then $C - B$ would include the toaster, which is not, however, really part of the strict lower contour set. This points to a possible idealisation which would preclude this kind of case: namely, that the desires people feel never leave them totally without a relative preference either way between two different things. Since, as discussed above, we are already abstracting away from the actual volatility in people's preferences, it is perhaps not unreasonable to add that idealisation. That is to say, without much compounding the extent of our idealisation of actual human behaviours, we may suppose that no–one will ever be in the situation of having not the least preference either way between two quite different things.

The preference relation $\succ_i$ is said to be

- ▷ **nongranular** if $\{(x, y) : x \succ_i y\}$ is an open subset of $X_i \times X_i$;
- ▷ **locally nonsatiated at** $x_i \in X_i$ if for each $\varepsilon > 0$ there exists $x_i' \in X_i$ such that $\|x_i - x_i'\| < \varepsilon$ and $x_i' \succ_i x_i$;
- ▷ **locally nonsatiated** (on X_i) if it is locally nonsatiated at each point of X_i.

Definition 2.2.2 *A metric space X is* ***locally compact*** *(respectively,* ***locally totally bounded****) if each bounded subset of X is contained in a compact (respectively, totally bounded) set.*

Theorem 1 of [20] says that if X_i is locally compact and $\succ_i$ is locally nonsatiated at each compact subset of X_i, then the strict upper contour set $(x, \rightarrow)$ is located in X_i for each $x \in X_i$; in that case, $(x, \rightarrow)$ is dense in the upper contour set $[x, \rightarrow)$, which is also located. If the preference relation $\succ_i$ is nongranular, then $[x, \rightarrow)$ is closed and hence locally compact.

By a **utility function** for (or representing) the preference relation $\succ_i$ we mean a mapping $u_i : X_i \rightarrow \mathbb{R}$ such that

$$\forall x, y \in X_i \ (x \succ_i y \Leftrightarrow u_i(x) > u_i(y)).$$

Note that in general there will be many functions that satisfy the stipulated condition. It is easily possible indeed for there to be embarrassingly many such functions. If people's preferences are few and not significantly structured there may be so many functions that satisfy the stipulated condition, that we would have to say that the preferences are without metrical structure, even though any specific utility function seems to imply that there is a metric. To switch from one permissible utility function to another would in that case not need to be metric–preserving. In such a case the very idea of a utility function is not all that natural, precisely because functions of the requisite sort are so thick on the ground. If, however, as many economists suppose, people's preferences are really very well structured (say because people each implicitly put a specific dollar value on anything), then the utility functions that are admissible given the above stipulation would differ from one another just by a scale factor and possibly a translation. Even in this case there would be many admissible functions. To the extent that people's preferences are rationally structured, it is therefore inevitable that there will be utility functions.

Thus while Pareto himself sought to replace utility considerations with considerations of another type, we need only the concepts on which his own reasoning is based in order to introduce the concept of utility after all. Once we have done so, we may become tempted to compare, on the basis of their aggregate utility, states that are not strictly comparable using Pareto's own considerations. For example suppose that there is no way that a bomb disposal expert can save a thousand people from imminent destruction without scratching his finger. Suppose for the sake of argument that he regards himself worse off if he scratches his finger than if he does not. Then we have two states neither of which is Pareto optimal, since in the one case the bomb disposal expert is worse off for having scratched his finger, and in the other case a thousand people are worse off for having lost their lives. Here there seem to be overwhelming reasons relating to utility to prefer the former state to the latter, even though those two states are not Pareto rankable. Thus once we are armed with a concept of utility we will feel tempted to make such further comparisons.

In the present discussion, the utility notion just introduced will not be used in this way. It will be used strictly in support of Pareto–like considerations. We may note in passing that utility can sometimes seem a poor basis upon which to compare alternative states. While sometimes utility considerations seem to make an overwhelming case for the preferability of one to another of two states that cannot be Pareto compared, often that kind of case is not at all strong. It can even seem outright wrong, as when considerations of overall utility recommend a kind of tyranny of the majority.

Let us make one further remark, an important one for the mathematical development later on. We will follow economists generally in assuming that the preference structures out there in the real world are exceedingly rich. Indeed, we will assume, as is completely standard, that they are so rich that the utility functions that are admissible as such by the above criterion inevitably form a very narrow family, so that we may assume of any utility function u_k that it is continuous. As remarked in Chapter 1, some considerable idealisation of economic behaviours by real people is required in support of this assumption. Such idealisation is justified because of the mathematical elegance that it alone can bring. Yet, when constructivists assume continuity, as we do here, they employ an idealisation that is in a sense infinitely more modest than that which a classical mathematician would employ. We make no recourse to actual rather than merely potential infinities; we make no assumptions that there are infinities of different sizes; and so on.

Let $\xi_i \in X_i$, and let p be a unit vector in $\mathbb{R}^N$. We say that ξ_i is a **chosen point for consumer** i under the price vector p if

$$\forall x \in X_i \ (x \succ_i \xi_i \Rightarrow \langle p, x \rangle > \langle p, \xi_i \rangle) .$$

In that case we have

$$\forall x \in X_i \left(\langle p, \xi_i \rangle \geq \langle p, x \rangle \Rightarrow \xi_i \not\prec x\right).$$

We might reasonably expect a chosen point to satisfy the following **revealed preference condition:**

$$\forall x \in X_i \ (\langle p, \xi_i \rangle > \langle p, x \rangle \Rightarrow \xi_i \succ_i x) .$$

(If ξ_i is both chosen under the price p and costs more than the consumption vector $x \in X_i$, then ξ_i is strictly preferred to x_i.)

Lemma 2.2.3 *If $\succ_i$ is locally nonsatiated and $\xi_i \in X_i$ is a chosen point for consumer i under the price vector p, then ξ_i satisfies the revealed preference condition.*

Proof. Let $x \in X_i$ and $\langle p, \xi_i \rangle > \langle p, x \rangle$. By the continuity of the mapping $\tau : \mathbb{R}^N \to \mathbb{R}$ defined by $\tau(\xi) = \langle p, \xi \rangle$, there exists $r > 0$ such that if $x' \in X$ and $\|x' - x\| < r$, then $\langle p, \xi_i \rangle > \langle p, x' \rangle$. As $\succ$ is locally nonsatiated at x, there exists $x' \in X$ such that $x' \succ x$ and $\|x' - x\| < r$. Then $\langle p, \xi_i \rangle > \langle p, x' \rangle$, so $\xi \not\prec_i x'$; whence $\xi_i \not\prec x' \succ x$ and therefore $\xi_i \succ x$. *q.e.d.*

We say that a point ξ_i of X_i is an ε–**almost chosen point** for consumer i under the price vector p if

$$\forall x \in X_i \ (x \succ_i \xi_i \Rightarrow \langle p, x\rangle > \langle p, \xi_i\rangle - \varepsilon).$$

If ξ_i is a chosen point under p, then for each $\varepsilon > 0$ it is an ε–almost chosen point under p.

Lemma 2.2.4 *Let ξ_i be an ε–almost chosen point for consumer i under p, and suppose that $\succ_i$ is locally nonsatiated. If $x \in X_i$ and $\langle p, \xi_i\rangle > \langle p, x\rangle + \varepsilon$, then $\xi_i \succ_i x$.*

Proof. Let $x \in X_i$ satisfy $\langle p, \xi_i\rangle > \langle p, x\rangle + \varepsilon$. Choose $r > 0$ such that if $x' \in X_i$ and $\|x' - x\| < r$, then $\langle p, \xi_i\rangle > \langle p, x'\rangle + \varepsilon$. Then choose $x' \in X_i$ such that $\|x' - x\| < r$ and $x' \succ_i x$. Either $x' \succ_i \xi_i$ or $\xi_i \succ_i x$. In the former case we have $\langle p, x'\rangle > \langle p, \xi_i\rangle - \varepsilon$ and therefore $\langle p, \xi_i\rangle < \langle p, x'\rangle + \varepsilon$, a contradiction. Hence that case is ruled out and we must have $\xi_i \succ_i x$. *q.e.d.*

2.3 Approximate Equilibrium 1 Implies Approximate Pareto Optimum

Let $\xi \in X_1 \times \cdots \times X_m$, let $\eta \in Y_1 \times \cdots \times Y_n$, and let p be a unit vector in $\mathbb{R}^N$. We say that the triple (ξ, η, p) is an **equilibrium** if the following three conditions obtain:

E1 For each i, ξ_i is a chosen point under p;

E2 For each j, if $y_j \in Y_j$, then $\langle p, \eta_j\rangle \geq \langle p, y_j\rangle$;

E3 $\sum\limits_{i=1}^{m} \xi_i = \sum\limits_{j=1}^{n} \eta_j + \bar{x}$.

Given $\varepsilon > 0$, we say that the triple (ξ, η, p) is an ε–**approximate equilibrium 1** if the following conditions hold:

εE1 For each i, ξ_i is an ε–almost chosen point under the price vector p;

εE2 For each j and each $y_j \in Y_j$, $\langle p, \eta_j \rangle > \langle p, y_j \rangle - \varepsilon$;

εE3 $\sum\limits_{i=1}^{m} \xi_i = \sum\limits_{j=1}^{n} \eta_j + \bar{x}$.

We say that a feasible array $\xi = (\xi_1, \ldots, \xi_m)$ of consumption vectors is a **Pareto optimum** if for each feasible array $\mathbf{x}$ of consumption vectors such that $x_i \succ_i \xi_i$ for some i, there exists k such that $\xi_k \succ_k x_k$. Suppose also that each $\succ_i$ is represented by a utility function $u_i : X_i \to \mathbb{R}$; we say that ξ is an ε–**approximate Pareto optimum** if for each feasible array $\mathbf{x}$ of consumption vectors such that $x_i \succ_i \xi_i$ for some i, then $u_k(\xi_k) > u_k(x_k) - \varepsilon$ for some k. Note that in the presence of a utility function, ξ is a Pareto optimum if and only if it is an ε–approximate Pareto optimum for each $\varepsilon > 0$.

> *From now on we assume that each consumption set X_i is locally compact and that each preference relation $\succ_i$ is represented by a utility function $u_i : X_i \to \mathbb{R}$.*

Bearing in mind Lemma 2.2.3, we say that $\xi_i \in X_i$ is a (u_i, ε)–**chosen point** for consumer i under the price vector p, where $\varepsilon > 0$, if there exists $\alpha > 0$ such that

$$\forall x \in X_i \; (\langle p, \xi_i \rangle \geq \langle p, x \rangle - \alpha \Rightarrow u_i(\xi_i) \geq u_i(x) - \varepsilon).$$

Classically, if X_i is compact, u_i is continuous, and ξ_i is a chosen point under p, then for each $\varepsilon > 0$, ξ_i is a (u_i, ε)–chosen point under p. Indeed, suppose that ξ_i is chosen but, for some $\varepsilon > 0$, not (u_i, ε)–chosen. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in X_i such that $\langle p, \xi_i \rangle \geq \langle p, x_n \rangle - n^{-1}$ and $u_i(\xi_i) < u_i(x_n) - \varepsilon$ for each n. Without loss of generality, we may assume that $(x_n)_{n=1}^{\infty}$ converges to a limit in the compact space X_i. Then $\langle p, \xi_i \rangle \geq \langle p, x \rangle$; moreover, $u_i(x) \geq u_i(\xi_i) + \varepsilon$ and therefore $x \succ_i \xi_i$, which is absurd, since ξ_i is chosen.

We say that a feasible array ξ of consumption vectors is an ε**–threshold Pareto optimum** if for each feasible array $\mathbf{x}$ of consumption vectors such that $u_i(x_i) > u_i(\xi_i) + \varepsilon$ for some i, there exists k such that $\xi_k \succ_k x_k$. (Thus the difference between the utilities of x_i and ξ_i has to exceed the threshold ε before we can guarantee that for some k, x_k is strictly preferred to ξ_k.)

Theorem 2.3.1 *Let ξ be an array of consumption vectors, let η be an array of production vectors such that $\sum_{i=1}^{m} \xi_i = \sum_{j=1}^{n} \eta_j + \bar{x}$, let p be a unit vector in $\mathbb{R}^N$, and let $\varepsilon > 0$. Suppose that for each i and under the price vector p, ξ_i is an (u_i, ε)–chosen point for consumer i and satisfies the revealed preference condition. Then there exists $\delta > 0$ with the following property: if $\langle p, \eta_j \rangle \geq \langle p, y_j \rangle - \delta$ for each j and each $y_j \in Y_j$, then ξ is an ε–threshold Pareto optimum.*

Proof. Choose $\alpha > 0$ such that for each i $(1 \leq i \leq m)$, if $x_i \in X_i$ and $\langle p, \xi_i \rangle > \langle p, x_i \rangle - \alpha$, then $u_i(\xi_i) > u_i(x_i) - \varepsilon$. Setting $\delta = \frac{\alpha}{n}$, suppose that $\langle p, \eta_j \rangle \geq \langle p, y_j \rangle - \delta$ for each j and each $y_j \in Y_j$. Without loss of generality, consider a feasible array $(x_1, \ldots, x_m)$ of consumption vectors such that $u_1(x_1) > u_1(\xi_1) + \varepsilon$. Then, by our choice of α, $\langle p, x_1 \rangle \geq \langle p, \xi_1 \rangle + \alpha$. With $(y_1, \ldots, y_n)$ an admissible array of production vectors such that

$$\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j + \bar{x},$$

we have

$$\begin{aligned}
\sum_{i=1}^{m} \langle p, x_i \rangle &= \sum_{j=1}^{n} \langle p, y_j \rangle + \langle p, \bar{x} \rangle \\
&< \sum_{j=1}^{n} \left(\langle p, \eta_j \rangle + \frac{\alpha}{n} \right) + \langle p, \bar{x} \rangle \\
&= \sum_{i=1}^{m} \langle p, \xi_i \rangle + \alpha \\
&\leq \langle p, x_1 \rangle + \sum_{i=2}^{m} \langle p, \xi_i \rangle .
\end{aligned}$$

Thus

$$\sum_{i=2}^{m} (\langle p, \xi_i \rangle - \langle p, x_i \rangle) > 0,$$

and so there exists $k \geq 2$ such that $\langle p, \xi_k \rangle > \langle p, x_k \rangle$. It follows from the revealed preference condition that $\xi_k \succ x_k$. *q.e.d.*

2.4 Approximate Pareto Optimum Implies Approximate Equilibrium 1

Our next aim is to establish partial converses of some of the foregoing results. For this we need some more definitions and preliminary results.

A subset X of $\mathbb{R}^N$ is said to be **bounded below** if there exists $c < 0$ such that for each $x = (x_1, \ldots, x_N) \in X$ and for each k, we have $x_k \geq c$. We then say that X is **bounded below by** c. Informally, to say that the consumption set X_i is bounded below means that there is a bound to the amount of any good/service that consumer i can offer in return for some more desirable good/service.

Lemma 2.4.1 *Suppose that each X_i is bounded below by $c < 0$, and let M be a positive number. If $x_i \in X_i$ for each i, and $\left\| \sum_{i=1}^{m} x_i \right\| < M$, then*

$$\max_{1 \leq i \leq m} \|x_i\| \leq M - mc. \tag{2.1}$$

Proof. Writing $x_i = (x_1^i, \ldots, x_N^i)$, suppose that

$$\|x_i\| = \max \left\{ \left|x_1^i\right|, \left|x_2^i\right|, \ldots, \left|x_N^i\right| \right\} > M - mc.$$

Choose j such that $\left|x_j^i\right| > M - mc$. Then $\left|x_j^i\right| > -mc \geq -c > 0$; so as $x_j^i \geq c$, we must have

$$x_j^i = \left|x_j^i\right| > M - mc.$$

Thus

$$\begin{aligned} x_j^1 + x_j^2 + \cdots + x_j^i + \cdots + x_j^m &> c + c + \cdots + (M - mc) + c + \cdots + c \\ &= M - mc + (m-1)\,c \\ &= M - c \end{aligned}$$

and therefore

$$\left\| \sum_{i=1}^{m} x_i \right\| = \max \left\{ \left| \sum_{i=1}^{m} x_1^i \right|, \ldots, \left| \sum_{i=1}^{m} x_N^i \right| \right\} > M - c > M,$$

which contradicts our hypotheses. Hence (2.1) obtains. *q.e.d.*

We now state Ishihara's strengthening of Bishop's separation theorem ([9], page 336, (4.3); [59]). Note that Ishihara's theorem requires the additional hypotheses of locatedness which holds automatically under classical logic.

Theorem 2.4.2 *Let H be a Hilbert space, and C a convex subset of H that is* ***located*** *in the sense that*

$$\rho(x, C) = \inf \{\|x - y\| : y \in C\}$$

exists for each $x \in H$. Suppose that $\rho(0, C) > 0$. Then there exists a vector $p \in H$ of norm 1 *such that $\langle x, p \rangle \geq \rho(0, C)$ for each $x \in C$.*

From a classical point of view the assumption that there is a well–defined distance between an element $x \in H$ and a convex subset C of H comes as it were for free. From a constructive point of view by contrast, something substantive hinges on this property of locatedness, viz.: that the distances in question are computable. Our results here therefore tell something that the classical mathematician would quite fail to remark.

Now suppose that X_i is a convex subset of $\mathbb{R}^N$. We say that the preference relation $\succ_i$ is

- $\triangle$ **quasi–strictly convex** if for any two distinct $x, y \in X$ the relations $x \not\prec_i \xi$ and $y \not\prec_i \xi$ together imply that $tx + (1-t)\, y \succ_i \xi$ for all $t \in (0,1)$;

- $\triangle$ **strictly convex 1** if for any two distinct $x, y \in X$ and for each $t \in (0,1)$, either $tx + (1-t)\, y \succ_i x$ or $tx + (1-t)\, y \succ_i y$.

Although strict convexity 1 implies quasi–strict convexity, and classically these two notions are equivalent, they are not equivalent constructively. However, it follows from Corollary 1.2 of [18] that if X_i is complete and $\succ_i$ is locally nonsatiated and quasi–strictly convex, then $\succ_i$ is strictly convex 1.

The proof of our next theorem is based on the standard classical idea of applying a functional–analytic separation theorem, as found on pages 195–198 of [84].

Theorem 2.4.3 *Let each X_i be convex, locally compact, and bounded below, and let Y be convex and compact. Suppose that for each i $(1 \leq i \leq m)$,*

(i) *the preference relation $\succ_i$ is quasi–strictly convex and locally nonsatiated;*

(ii) *the strict upper contour set $(\xi_i, \rightarrow)$ is open in $\mathbb{R}^N$.*

Let ξ be an array of consumption vectors and η an array of production vectors such that $\sum_{i=1}^{m} \xi_i = \sum_{j=1}^{n} \eta_j + \bar{x}$, and let $\varepsilon > 0$. If ξ is an $\varepsilon/2$–threshold Pareto optimum, then there exist a unit vector $p \in \mathbb{R}^N$ and a positive number δ such that

$$\forall i \, \forall x_i \in X_i \; (u_i(x_i) \geq u_i(\xi_i) + \varepsilon \Rightarrow \langle p, x_i \rangle \geq \langle p, \xi_i \rangle + \delta).$$

Proof. Let

$$A = \left\{ \sum_{i=1}^{m} x_i : \exists k \; (u_k(x_k) \geq u_k(\xi_k) + \varepsilon \wedge \forall i \neq k \; (x_i = \xi_i)) \right\}$$

and

$$B = \left\{ x \in \mathbb{R}^N : \exists y \in Y \; (x = y + \bar{x}) \right\}.$$

Clearly, B is convex. To prove A convex, we first observe that, by the remark preceding the statement of Theorem 2.4.3, $\succ_k$ is strictly convex 1. Consider two elements $\sum_{i=1}^{m} x_i$ and $\sum_{i=1}^{m} x_i'$ of A. If $x_k \neq x_k'$, then for each $t \in (0,1)$ either $tx_k + (1-t)\, x_k' \succ_k x_k$ and therefore

$$u_k(tx_k + (1-t)\, x_k') > u_k(x_k) \geq u_k(\xi_k) + \varepsilon,$$

or else $tx_k + (1-t)\, x_k' \succ_k x_k'$, and similarly,

$$u_k(tx_k + (1-t)\, x_k') \geq u_k(\xi_k) + \varepsilon. \tag{2.2}$$

It follows from the continuity of u_k that (2.2) holds for all $t \in [0,1]$. In the general case, where we may not know whether $x_k = x_k'$ or $x_k \neq x_k'$, suppose that for some $t \in [0,1]$ we have

$$u_k(tx_k + (1-t)\, x_k') < u_k(\xi_k) + \varepsilon \leq u_k(x_k).$$

Then, by the continuity of u_k,

$$tx_k + (1-t)\, x_k' \neq x_k = tx_k + (1-t)\, x_k;$$

whence $(1-t)(x_k - x_k') \neq 0$ and therefore $x_k \neq x_k'$. The previous case now shows that (2.2) holds, which is a contradiction. Hence, in fact, (2.2) holds in the general case.

We next show that $A - B$ is located. Since X_k is locally compact and u_k is continuous, we may assume without loss of generality that

$$S_k = \{x_k \in X_k : u_k(x_k) \geq u_k(\xi_k) + \varepsilon\},$$

is locally compact in $\mathbb{R}^N$ ([9], page 98, Theorem (4.9)) and therefore that

$$A_k = \{\xi_1\} \times \cdots \times \{\xi_{k-1}\} \times S_k \times \{\xi_{k+1}\} \times \cdots \times \{\xi_m\}$$

is a locally compact subset of $\mathbb{R}^N \times \cdots \times \mathbb{R}^N$ (m factors). It follows that

$$Z = \left(\bigcup_{k=1}^{m} A_k\right) \times Y$$

is a locally totally bounded subset of $(\mathbb{R}^N \times \cdots \times \mathbb{R}^N) \times \mathbb{R}^N$. (Note that the union of two closed subsets of a metric space may not be provably closed in constructive mathematics.) Define a uniformly continuous mapping Φ of Z onto $A - B$ by

$$\Phi(\mathbf{x}, y) = \sum_{i=1}^{m} x_i - y - \bar{x}.$$

Choose $c < 0$ such that each X_i is bounded below by c, and $b > 0$ such that $\|y\| \leq b$ for all $y \in Y$. Let S be a bounded subset of $A - B$. Choose $R > 0$ such that $\|s\| \leq R$ for all $s \in S$. If $(\mathbf{x}, y) \in \Phi^{-1}(S)$, then

$$\left\|\sum_{i=1}^{m} x_i\right\| < \|\Phi(\mathbf{x}, y)\| + \|y\| + \|\bar{x}\| \leq R + b + \|\bar{x}\|$$

and so, by Lemma 2.4.1,

$$\max_{1 \leq i \leq m} \|x_i\| \leq R + b + \|\bar{x}\| - mc.$$

Hence

$$\begin{aligned}
\|(\mathrm{x}, y)\| &= \max\{\|\mathrm{x}\|, \|y\|\} \\
&= \max\{\|x_1\|, \ldots, \|x_m\|, \|y\|\} \\
&\leq \max\{R + b + \|\bar{x}\| - mc, b\} \\
&= R + b + \|\bar{x}\| - mc.
\end{aligned}$$

Hence $\Phi^{-1}(S)$ is a bounded subset of Z. Since Z is locally compact, there exists a compact set $T \subset Z$ such that $\Phi^{-1}(S) \subset T$ and therefore $S \subset \Phi(T)$. But Φ is uniformly continuous, so $\Phi(T)$ is a totally bounded subset of $A - B$. Thus $A - B$ is locally totally bounded and hence located ([38], page 33, Theorem (4.11)).

Classically, in order to apply an appropriate separation theorem, not only it is unnecessary to discuss the locatedness of $A - B$, but also it is enough to show that $0 \notin A - B$. Constructively, we must do a lot more than the latter: we must show that $\rho(0, A - B) > 0$. It is to that end that we introduced hypothesis (ii). Suppose, then, that ξ is an $\varepsilon/2$–threshold Pareto optimum. Using hypothesis (ii) and the continuity of the utility functions, choose $r \in (0, 1)$ such that for $1 \leq k \leq m$, if

$$x_k \in X_k,\ \|x_k\| \leq 1 + b + \|\bar{x}\| - mc + r,\ u_k(x_k) \geq u_k(\xi_k) + \varepsilon,$$

and $\|x'_k - x_k\| < r$, then

$$x'_k \in X_k \text{ and } u_k(x'_k) \geq u_k(\xi_k) + \varepsilon/2. \tag{2.3}$$

Either $\rho(0, A - B) > 0$ or, as we may assume, $\rho(0, A - B) < 1/2$ and therefore

$$\rho(0,\ A - B) = \rho\left(0, (A - B) \cap \overline{B}(0, 1)\right).$$

Consider any k, any element x of A_k, and any $y \in Y$ such that

$$\left\| \sum_{i=1}^{m} x_i - y - \bar{x} \right\| < 1.$$

Then

$$\left\| \sum_{i=1}^{m} x_i \right\| \le 1 + \|y\| + \|\bar{x}\| \le 1 + b + \|\bar{x}\|,$$

so by Lemma 2.4.1,

$$\max_{1 \le i \le m} \|x_i\| \le 1 + b + \|\bar{x}\| - mc.$$

Let

$$x'_k = - \sum_{i=1,\, i \ne k}^{m} \xi_i + y + \bar{x}.$$

Either $\|x_k - x'_k\| > r/2$ or $\|x_k - x'_k\| < r$. In the latter case,

$$\|x'_k\| \le \|x_k\| + \|x_k - x'_k\| < 1 + b + \|\bar{x}\| - mc + r$$

and so (2.3) holds, by our choice of r; but this is impossible: for since

$$x'_k + \sum_{i=1, i \ne k}^{m} \xi_i = y + \bar{x},$$

the consumption vector $(\xi_1, \ldots, \xi_{k-1}, x'_k, \xi_{k+1}, \ldots, \xi_m)$ is feasible, $x'_k \succ_k \xi_k$, and ξ is an $\dot{\varepsilon}/2$–threshold Pareto optimum. Hence the case $\|x_k - x'_k\| < r$ is ruled out. Since $\sum_{i=1}^{m} x_i - y - \bar{x}$ is an arbitrary element of $(A - B) \cap \overline{B}(0, 1)$, we conclude that $\rho(0, A - B) > r/2$.

Now let

$$\delta = \rho(0, A - B) > 0.$$

Applying Theorem 2.4.2 with $C = A - B$, we obtain a unit vector $p \in \mathbb{R}^N$ such that

$$\langle p, x_1 + x_2 + \cdots + x_m \rangle \ge \langle p, y + \bar{x} \rangle + \delta$$

whenever $\mathrm{x} \in A_k$ and $y \in Y$. For such x, since $\sum_{i=1}^{m} \xi_i \in B$, we have

$$\langle p, \xi_1 + \xi_2 + \cdots + x_k + \xi_{k+1} + \cdots + \xi_m \rangle \ge \left\langle p, \sum_{i=1}^{m} \xi_i \right\rangle + \delta$$

and therefore $\langle p, x_k \rangle \ge \langle p, \xi_k \rangle + \delta$. *q.e.d.*

Corollary 2.4.4 *Under the hypotheses of Theorem 2.4.3, if ξ is an $\frac{\varepsilon}{2}$–threshold Pareto optimum, then there exists a unit vector $p \in \mathbb{R}^N$ under which ξ_k is an (u_k, ε)–chosen point for each k $(1 \leq k \leq m)$.*

Proof. Construct δ as in Theorem 2.4.3. If $x_k \in X_k$ and $\langle p, \xi_k \rangle > \langle p, x_k \rangle - \delta$, then $\neg (u_k(x_k) \geq u_k(\xi_k) + \varepsilon)$ and so $u_k(\xi_k) \geq u_k(x_k) - \varepsilon$. *q.e.d.*

Condition (ii) of Theorem 2.4.3 requires further comment. Classically, the continuity of the utility function u_i ensures that $(\xi_i, \rightarrow)$ is open in X_i; but we adopt the stronger hypothesis that $(\xi_i, \rightarrow)$ is open in $\mathbb{R}^N$, which is classically equivalent to saying that no boundary point of X_i is strictly preferred to ξ_i. We used this stronger hypothesis to prove that $\rho(0, A - B) > 0$, in order to set up our application of the separation theorem (Theorem 2.4.2).

The preference relation $\succ_i$ on X_i is said to be **convex** if the following three conditions hold:

- $\bigtriangledown$ X_i is convex;
- $\bigtriangledown$ If $x \succ_i \xi$ and $x' \succ_i \xi$, then $tx + (1-t)x' \succ_i \xi$ whenever $0 \leq t \leq 1$;
- $\bigtriangledown$ If $x \not\prec_i \xi$ and $x' \not\prec_i \xi$, then $tx + (1-t)x' \not\prec_i \xi$ whenever $0 \leq t \leq 1$.

In that case the sets $[x, \rightarrow)$ and $(x, \rightarrow)$ are convex.

We next state a separation theorem whose constructive proof requires relatively routine modifications of the work on pages 383–384 of [2].

Definition 2.4.5 *The **complement** of a subset S of a metric space E is*

$$\sim S = \{x \in E : \forall y \in S \ (\rho(x, y) > 0)\}.$$

Theorem 2.4.6 *Let A and B be convex subsets of $\mathbb{R}^N$ such that $A - B$ is located and $0 \in \sim (A - B)$. Then for each $\varepsilon > 0$ there exist a nonzero vector $p \in \mathbb{R}^N$ and a real number c such that $\langle p, x \rangle \geq c$ for all $x \in A$, and $\langle p, x \rangle \leq c + \varepsilon$ for all $x \in B$.*

The following result leads along a slightly different path towards a constructive version of "Pareto optimum implies equilibrium".

Proposition 2.4.7 *Let each X_i be convex, locally compact, and bounded below, and let the aggregate production set Y be convex and compact. Let ξ be a Pareto optimum and let η be an admissible array of production vectors such that*

$$\sum_{i=1}^{m} \xi_i = \sum_{j=1}^{n} \eta_j + \bar{x}.$$

Suppose that for each i,

(i) *consumer i's preference relation $\succ_i$ is locally nonsatiated at each compact subset of X_i and convex;*

(ii) *the strict upper contour set $(\xi_i, \rightarrow)$ is open in $\mathbb{R}^N$.*

Then for each $\varepsilon > 0$ there exist a unit vector $p \in \mathbb{R}^N$ and a real number α such that

- *if $x_1 \succ_1 \xi_1$ and $x_i \succcurlyeq_i \xi_i$ for each $i \geq 2$, then $\left\langle p, \sum_{i=1}^{m} x_i \right\rangle \geq \alpha$;*
- *$\langle p, y + \bar{x} \rangle \leq \alpha + \varepsilon/2$ for each $y \in Y$.*

Moreover, $\left\langle p, \sum_{i=1}^{m} \xi_i \right\rangle \geq \alpha$.

Proof. We only sketch the proof, since it is fairly similar to that of Theorem 2.4.3. Define convex sets by

$$A = \left\{ x \in \mathbb{R}^N : \exists x_1 \succ_1 \xi_1 \wedge \forall i \geq 2\, \exists x_i \succcurlyeq_i \xi_i \left(x = \sum_{i=1}^{N} x_i \right) \right\}$$

and

$$B = \left\{ x \in \mathbb{R}^N : \exists y \in Y \ (x = y + \bar{x}) \right\}.$$

Applying Theorem 1 of [22], we see that

$$Z = (\xi_1, \rightarrow) \times [\xi_2, \rightarrow) \times [\xi_3, \rightarrow) \times \cdots \times [\xi_m, \rightarrow) \times Y$$

is a locally totally bounded subset of $\mathbb{R}^N \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N$ ($m+1$ factors). Define a uniformly continuous mapping Φ of Z onto $A - B$ by

$$\Phi(x_1, \ \ldots \ , x_m, y) = \sum_{i=1}^{m} x_i - y - \bar{x}.$$

An argument like that in the proof of Theorem 2.4.3 shows that if S is a bounded subset of $A - B$, then $\Phi^{-1}(S)$ is bounded; whence $A - B$ is locally totally bounded and therefore located. Again using ideas from the proof of Theorem 2.4.3, we can show that $0 \in \sim (A - B)$.

Given $\varepsilon > 0$, we now apply Theorem 2.4.6 to produce a unit vector $p \in \mathbb{R}^N$ and a real number α such that $\langle p, a \rangle \geq \alpha$ for all $a \in A$, and $\langle p, b \rangle \leq \alpha + \varepsilon/2$ for all $b \in B$. It remains to prove that $\left\langle p, \sum_{i=1}^{m} \xi_i \right\rangle \geq \alpha$. To this end, fix $x_1 \succ_1 \xi_1$, and for each $t \in (0,1)$ write

$$\begin{aligned} z_1(t) &= tx_1 + (1-t)\,\xi_1, \\ z_i(t) &= \xi_i \quad (2 \leq i \leq m)\,, \\ z(t) &= \sum_{i=1}^{m} z_i(t). \end{aligned}$$

The convexity of $\succ_1$ ensures that $z_1(t) \in (\xi_1, \rightarrow)$ and therefore $z(t) \in A$. Hence $\langle p, z(t) \rangle \geq \alpha$. Letting $t \rightarrow 0$ and using the continuity of the inner product, we see that $\left\langle p, \sum_{i=1}^{m} \xi_i \right\rangle \geq \alpha$. *q.e.d.*

Corollary 2.4.8 *Under the hypotheses of* Proposition 2.4.7, *for each* $\varepsilon > 0$ *there exists a price vector* p *such that* (ξ, η, p) *is an* ε*−approximate equilibrium 1.*

Proof. Given $\varepsilon > o$ let A, B, p, α be as in the proof of Proposition 2.4.7. Since

$\sum_{i=1}^{m} \xi_i \in B$, we have

$$\alpha \leq \left\langle p, \sum_{i=1}^{m} \xi_i \right\rangle \leq \alpha + \varepsilon/2.$$

It follows that if $x_1 \succ_1 \xi_1$, then, as $x_1 + \xi_2 + \cdots + \xi_m \in A$,

$$\langle p, x_1 + \xi_2 + \cdots + \xi_m \rangle \geq \alpha \geq \left\langle p, \sum_{i=1}^{m} \xi_i \right\rangle - \varepsilon/2$$

and therefore $\langle p, x_1 \rangle \geq \langle p, \xi_1 \rangle - \varepsilon/2 > \langle p, \xi_1 \rangle - \varepsilon$. Similarly, if $i \geq 2$ and $x_i \succ_i \xi_i$, then $\langle p, x_i \rangle \geq \langle p, \xi_i \rangle - \varepsilon$.

Finally, if $(y_1, \ldots, y_n)$ is an admissible array of production vectors, then $\sum_{j=1}^{n} y_j + \bar{x} \in B$, so

$$\left\langle p, \sum_{j=1}^{n} y_j + \bar{x} \right\rangle \leq \alpha + \varepsilon/2 \leq \left\langle p, \sum_{i=1}^{m} \xi_i \right\rangle + \varepsilon = \left\langle p, \sum_{i=1}^{m} \eta_i + \bar{x} \right\rangle + \varepsilon$$

and therefore

$$\left\langle p, \sum_{j=1}^{n} y_j \right\rangle \leq \left\langle p, \sum_{i=1}^{m} \eta_i \right\rangle + \varepsilon.$$

Given $j \in \{1, \ldots, n\}$, and taking $y_j \in Y_j$ and $y_k = \eta_k$ for all $k \neq j$, we obtain $\langle p, \eta_j \rangle \geq \langle p, y_j \rangle - \varepsilon$. *q.e.d.*

What do we need to turn the approximate equilibria produced in Corollary 2.4.8 into an exact equilibrium? We argue classically as follows. For each positive integer n choose a unit price vector $p_n \in \mathbb{R}^N$ such that if $\mathbf{x}$ is feasible, then

$$\forall i \left(x_i \succ_i \xi_i \Rightarrow \langle p_n, x_i \rangle \geq \langle p_n, \xi_i \rangle - n^{-1} \right),$$

and if $\mathbf{y}$ is admissible, then

$$\forall j \left(\langle p_n, \eta_j \rangle \geq \langle p_n, y_j \rangle - n^{-1} \right).$$

Since the unit sphere of $\mathbb{R}^N$ is compact, by passing to a subsequence we may assume that the sequence $(p_n)_{n=1}^{\infty}$ converges to a unit vector $p \in \mathbb{R}^N$. It follows by the continuity of the inner product, if $\mathbf{x}$ is feasible, then

$$\forall i\ (x_i \succ_i \xi_i \Rightarrow \langle p, x_i \rangle \geq \langle p, \xi_i \rangle),$$

and if $\mathbf{y}$ is admissible, then

$$\forall j\ (\langle p, \eta_j \rangle \geq \langle p, y_j \rangle).$$

We can now push through the proof of the following result, as on pages 197–198 of [84] (or pages 308–309 of [20]):

> *Under the hypotheses of* Proposition 2.4.7, *suppose also that the following* **cheaper point condition** *holds: For each price vector p and each i $(1 \leq i \leq m)$, there exists $\xi_i' \in X_i$ such that $\langle p, \xi_i' \rangle < \langle p, \xi_i \rangle$. Then (ξ, η, p) is an equilibrium.*

The foregoing gives some idea of how much we can say about the connection between Pareto optima and equilibria using intuitionistic logic, and exactly where we appear to need the essentially nonconstructive notion of sequential compactness in order to progress from approximate to exact equilibria. Surely however we can query whether an economist will ever actually need more than the constructivist can give, here. It seems unnecessary for an economist to shoulder the full weight of the classical mathematician's idealising assumptions. Such assumptions implicate nonconstructive thinking about infinity, all of which lies infinitely far beyond the purview of practical, empirical evidence. Empirically determining the prices that real people have paid or sought for real products in the marketplace can hardly tell us whether or not to employ in our economic thinking the essentially nonconstructive notion of sequential compactness. Thus in a certain sense "exactness" here is a non–sequitur. It doesn't for a minute follow from the evidence that the essentially nonconstructive notion of sequential compactness should be used. By following the constructive approach we have employed more modest assumptions and nonetheless proved that the concepts of Approximate Equilibrium 1 and Approximate Pareto Optimum are equivalent. We believe that from an empirical standpoint an economist who employs these notions sacrifices nothing.

Chapter 3

Crossing the Boundary of a Convex Set in $\mathbb{R}^N$

3.1 Introduction

It is well–known that the full form of the intermediate value theorem cannot be proved constructively (although there are hypothesis that can be added to the usual ones, and that apply to most functions of interest in elementary analysis, in order to find the exact point at which the intermediate value is attained). A consequence of this is that, as we showed in Chapter 1, we cannot be sure of finding the point where the segment joining a point inside a convex subset Y of a Banach space to one outside Y crosses the boundary ∂Y, even if Y is **located** in the sense given in Theorem 2.4.2.

If Y is a closed, located and convex subset of $\mathbb{R}^N$, then it seems intuitively clear that there is a unique point $h(z)$ at which the line segment $[\xi, z]$ intersects ∂Y. The Corollary to Proposition 8 of [40] shows that the distance from that segment to ∂Y is 0; but that is not sufficient to establish constructively that the segment actually intersects ∂Y. Our aim in this paper is to show that in this case the segment really does meet the boundary, and that the resulting mapping h, taking z outside Y to the unique point of $[\xi, z] \cap \partial Y$, is continuous. Specifically, we prove:

Theorem 3.1.1 *Let Y be a closed located convex subset of $\mathbb{R}^N$, and ξ an interior point of Y. Then for each $z \in \sim Y^\circ$ the segment $[\xi, z]$ meets ∂Y in a unique point $h(z)$. Moreover, the mapping $h : \sim Y^\circ \rightarrow \partial Y$ is continuous.*

Definition 3.1.2 *The **metric complement** of S is*

$$-S = \{x \in X \exists r > 0 \forall s \in S \, (\rho(x, s) \geq r)\}$$

If S is located, then

$$-S = \{x \in X : \rho(x, S) > 0\}.$$

The proof of Theorem 3.1.1 requires a number of technical lemmas, which we develop shortly. Before doing so, though, we make two comments.

First, it is remarkable that the convexity and locatedness of Y suffice to provide the strong computational information that h is not only defined but also continuous on $\sim Y^\circ$: one might reasonably have expected that extra numerical information, such as that provided by the property of uniform convexity, would have been needed to establish the continuity of h.

Secondly, a word about the origins of our theorem, which lie in mathematical economics. In McKenzie's classical proof of the existence of an economic equilibrium ([84], pages 265–274), which we shall analyse in Chapter 5, the existence and continuity of the function h defined in Theorem 3.1.1 are essential for the construction of a set–valued function whose fixed points, produced by Kakutani's fixed–point theorem, are the desired equilibria. Our Theorem 3.1.1, which we believe is of geometrical interest in its own right, is a step towards a constructive analogue of McKenzie's proof.

3.2 Existence of Boundary Crossings

We now have a series of lemmas designed to lead to a proof of the existence of the boundary crossing mapping.

Lemma 3.2.1 *Let Y be a located, convex subset of $\mathbb{R}^N$, let $\xi \in Y^\circ$, and let $r > 0$ be such that $B(\xi, r) \subset Y$. Let $z \neq \xi$, $0 < t < 1$, and $z' = t\xi + (1-t)z$. If the ball $B(z, tr)$ intersects Y, then $B(z', t^2 r) \subset Y$.*

Proof. Suppose that there exists y in $B(z, tr) \cap Y$. Let $\zeta' \in B(z', t^2 r)$ and

$$\xi' = \left(1 - \frac{1}{t}\right) y + \frac{1}{t}\zeta'.$$

Then

$$\begin{aligned}
\|\xi - \xi'\| &= \left\| \left(\left(1 - \frac{1}{t}\right) z + \frac{1}{t} z' \right) - \left(\left(1 - \frac{1}{t}\right) y + \frac{1}{t}\zeta' \right) \right\| \\
&\leq \left(\frac{1}{t} - 1\right) \|z - y\| + \frac{1}{t} \|z' - \zeta'\| \\
&< \left(\frac{1}{t} - 1\right) tr + \frac{1}{t} t^2 r \\
&= r.
\end{aligned}$$

Hence $\xi' \in Y$. Since $\zeta' = t\xi' + (1-t)y$, it follows by convexity that $\zeta' \in Y$. *q.e.d.*

From now on we assume that Y is a closed, located, convex subset of $\mathbb{R}^N$ and that $\xi \in Y^\circ$.

Lemma 3.2.2 *If $z \neq \xi$, $0 < t < 1$, and $z' = t\xi + (1-t)z$, then either $z \in -Y$ or $z' \in Y^\circ$.*

Proof. Choose $r > 0$ such that $B(\xi, r) \subset Y$. Since Y is located, so is $-Y$ ([74], page 244, Proposition (1.5)). Let $s = \rho(z', -Y)$. Either $s > 0$ or $s < t^2 r$. In the first case, if $\|x - z'\| < s$, then $\rho(x, -Y) > 0$, so $\rho(x, Y) = 0$ and x belongs to the closed set Y. Hence $B(z', s) \subset Y$ and therefore $z' \in Y^\circ$. In the case $s < t^2 r$, we see from Lemma 3.2.1 that $B(z, tr) \cap Y = \emptyset$, so $z \in -Y$. *q.e.d.*

Proposition 3.2.3 *For each* $z \in \sim Y^\circ$,

$$t_z = \inf \{t \in [0,1] : t\xi + (1-t)z \in Y\}$$

exists and is < 1.

Proof. Fix $z \in \sim Y^\circ$, and define

$$S = \{t \in [0,1] : t\xi + (1-t)z \in Y\}.$$

Given $\varepsilon > 0$, compute real numbers $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $t_{i+1} - t_i < \varepsilon/4$ for each i, and set

$$z_i = t_i\xi + (1 - t_i)\, z \quad (0 \leq i \leq n).$$

Using Lemma 3.2.2, we can find m $(0 < m \leq n-2)$ such that $z_k \in Y^\circ$ for $0 \leq k \leq m$, and such that if $m < n-2$, then $z_{m+2} \in -Y$. We prove that $\{t_0, \ldots, t_{m-1}\}$ is an ε–approximation to S. If $m = n-2$, this follows from the fact that $\{t_0, \ldots, t_{m-1}\}$ is an $\varepsilon/2$–approximation to $[0,1]$. On the other hand, in the case $m < n-2$, since $z_{m+1} \in -Y$, the convexity of S ensures that $S \subset [0, t_{m+1}]$; then $\{t_0, \ldots, t_{m-1}\}$ is an ε–approximation to $[0, t_{m+1}]$ and therefore to S. Since $\varepsilon > 0$ is arbitrary, we have now shown that S is a totally bounded subset of $[0,1]$. Hence $\inf S$ exists, by [9] (page 38, Corollary (4.4)).

Now observe that if

$$t = 1 - \frac{r}{2\,\|\xi - z\|},$$

then

$$t > 1 - \frac{r}{\|\xi - z\|},$$

so $\|\xi - (t\xi + (1-t)\,z)\| < r$ and therefore $t\xi + (1-t)\,z \in Y^\circ$. It follows that $t_z \leq t < 1$. *q.e.d.*

Since Y is closed, the function defined on $\sim Y^\circ$ by

$$h(z) = t_z\xi + (1 - t_z)z$$

maps $\sim Y^\circ$ into Y. Also, $h(z) \neq \xi$, as $t_z < 1$. We call h the **boundary crossing map** corresponding to the interior point ξ of Y.

Proposition 3.2.4 *For each $z \in \sim Y^\circ$, $h(z) \in \partial Y$.*

Proof. By the definition of $h\,(z)$ as an infimum, $h\,(z)$ belongs to the closure of Y. Suppose that

$$0 < s = \rho\,(h\,(z)\,, -Y)\,.$$

Then (cf. the proof of Lemma 3.2.2) the ball $B(h\,(z)\,, s)$ lies in Y; we can now find $t < t_z$ such that $t\xi + (1-t)\,z$ belongs to that ball and therefore to Y. This contradicts the definition of $h\,(z)$. Hence $s = 0$ and therefore $h\,(z)$ lies in the closure of $-Y$. *q.e.d.*

Lemma 3.2.5 *If $z \neq \xi$, $0 \leq t < t_z < 1$, and $x = t\xi + (1-t)\,z$, then $x \in Y^\circ$.*

Proof. Write

$$h(z) = s\xi + (1-s)\,x,$$

with

$$0 < s = \frac{t_z - t}{1 - t} \leq 1$$

Since $x \neq h(z)$, we can apply Lemma 3.2.2 with z and z' replaced by x and $h(z)$ respectively, to show that either $x \in Y^{\circ}$ or $h(z) \in -Y$. Since $h(z) \in \partial Y$, the latter alternative is ruled out. *q.e.d.*

Definition 3.2.6 *A mapping f between metric spaces is* ***strongly extensional*** *if $f(x) \neq f(y)$ entails $x \neq y$.*

Proposition 3.2.7 *The mappings t_z and h are strongly extensional on $\sim Y^{\circ}$.*

Proof. Let $z, z' \in \sim Y^{\circ}$. First assume that $t_z \neq t_{z'}$; we may assume without loss of generality that $t_z < t_{z'}$. Writing

$$x = t_{z'}\xi + (1 - t_{z'})\, z,$$

we see from Lemma 3.2.5 that $x \in Y^{\circ}$. Since $t_{z'}\xi + (1 - t_{z'})z' \in \partial Y$, we see that $x \neq t_{z'}\xi + (1 - t_{z'})z'$; so $(1 - t_{z'})\, z \neq (1 - t_{z'})\, z'$ and therefore $z \neq z'$. Thus the mapping t_z is strongly extensional.

Now assume that $h(z) \neq h(z')$. Then

$$(z - z') - t_z\,(z - z') + (t_z - t_{z'})\,(\xi - z') = 0,$$

so $z - z' \neq 0$ or $t_z\,(z - z') \neq 0$ or $(t_z - t_{z'})\,(\xi - z') \neq 0$. In the first and second cases we have $z \neq z'$. In the remaining case, $t_z \neq t_{z'}$ and hence, by the first part of the proof, $z \neq z'$. *q.e.d.*

3.3 Continuity of Boundary Crossings

The classical proof of the continuity of the boundary crossing map h given in [84] contains at least two serious nonconstructive arguments: it starts by supposing that

h is not continuous at some $z \in \sim Y^\circ$, and it then derives a contradiction; to do this, it uses the nonconstructive sequential compactness property of $[0,1]$. With care, we can use some of the ideas of the classical proof as the basis of a constructive proof of

Proposition 3.3.1 *h is sequentially continuous on $\sim Y^\circ$.*

For convenience, fix $z \in \sim Y^\circ$, a sequence $(z_n)_{n=1}^{\infty}$ in $\sim Y^\circ$ converging to z, and $\varepsilon > 0$. Write

$$t_n = \inf\left\{t \in [0,1] : t\xi + (1-t)z_n \in Y\right\}.$$

Since $\sim Y^\circ$ is closed, and therefore complete, in $\mathbb{R}^N$, we can apply Ishihara's result ([60], Lemma 2) to the strongly extensional function h, to show that

either $\exists N \, \forall n \geq N \; \|h(z_n) - h(z)\| < \varepsilon$

or $\forall n \, \exists k > n \; \|h(z_n) - h(z)\| > \varepsilon/2$.

We want to rule out the second of these alternatives; this will show that h is sequentially continuous at z. So assume that the second alternative holds. Passing to subsequences if necessary, we may further assume without loss of generality that $\|h(z_n) - h(z)\| > \varepsilon/2$ and $\|z - z_n\| < \varepsilon/8$ for all n. Then

$$\begin{aligned}
\frac{\varepsilon}{2} &< \|(t_n - t_z)\xi + (z_n - z) + t_z z - t_n z_n\| \\
&\leq \|(t_n - t_z)\xi + (z_n - z) + (t_z - t_n)z + t_n(z - z_n)\| \\
&\leq |t_n - t_z| \, \|\xi - z\| + (1 + t_n)\|z_n - z\| \\
&< |t_n - t_z| \, \|\xi - z\| + \frac{2\varepsilon}{8}.
\end{aligned}$$

Hence

$$\forall n \; (|t_n - t_z| > \delta > 0) \tag{3.1}$$

where

$$\delta = \frac{\varepsilon}{4\,\|\xi - z\|}.$$

We aim to derive a contradiction from (3.1). The first step is peculiar, in that it introduces LPO into our arguments.

Lemma 3.3.2 *Statement* (3.1) *implies* LPO.

Proof. Let $(a_n)_{n=1}^{\infty}$ be an increasing binary sequence. If $a_n = 0$, set $\zeta_n = z$; if $a_n = 1 - a_{n-1}$, set $\zeta_k = z_n$ for all $k \geq n$. Then $(\zeta_n)_{n=1}^{\infty}$ is a Cauchy sequence in the complete space $\sim Y^\circ$ and so converges to a limit ζ in $\sim Y^\circ$. Either $t_\zeta \neq t_z$ or $|t_\zeta - t_z| < \delta$. In the first case, since the mapping t_z is strongly extensional, we have $\zeta \neq z$; so there exists n such that $\zeta_n \neq z$ and therefore $a_n = 1$. In the case $|t_\zeta - t_z| < \delta$, if $a_n = 1 - a_{n-1}$ then $\zeta = z_n$ and $|\zeta - z| > \delta$, a contradiction; whence we must have $a_n = 0$ for all n. *q.e.d.*

Lemma 3.3.3 LPO *implies that for each binary sequence* $(a_n)_{n=1}^{\infty}$, *either* $a_n = 0$ *for all sufficiently large* n *else* $a_n = 1$ *for infinitely many* n.

Proof. Given a binary sequence $(a_n)_{n=1}^{\infty}$, we may assume, from LPO, that there exists n_1 such that $a_{n_1} = 1$. Set $\lambda_1 = 1$. Having found $\lambda_k \in \{0, 1\}$ and the positive integer n_k, we apply LPO to the sequence $(a_{n_k+1}, a_{n_k+2}, \dots)$ to show that either $a_n = 0$ for all $n > n_k$ or else there exists $n_{k+1} > n_k$ such that $a_{n_{k+1}} = 1$. In the first case, we set $\lambda_{k+1} = 1$ and $n_j = n_k$ for all $j > k$; in the second, we set $\lambda_{k+1} = 0$. This completes the inductive construction of a binary sequence $(\lambda_k)_{k=1}^{\infty}$ and an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $\lambda_1 = 0$,

- if $\lambda_k = 0$ then $n_k > n_{k-1}$ and $a_{n_k} = 1$,
- if $\lambda_k = 1$, then $n_k = n_{k-1}$ and $a_n = 0$ for all $n > n_k$.

Applying LPO to $(\lambda_k)_{k=1}^{\infty}$, we see that either $\lambda_k = 0$ for all k, in which case $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers such that $a_{n_k} = 1$ for each k; or else $\lambda_k = 1$ for some k, and $a_n = 0$ for all sufficiently large n. *q.e.d.*

Lemma 3.3.4 LPO *implies the Bolzano–Weierstraß Theorem.*

Proof. Assuming LPO, consider any sequence $(x_n)_{n=1}^{\infty}$ in $[0,1]$. By LPO, for each n either $x_n \leq 1/2$ or $x_n > 1/2$. Thus we can define a binary sequence $(a_n)_{n=1}^{\infty}$ such that

$$\begin{aligned} a_n = 0 \ &\Rightarrow\ x_n \leq 1/2, \\ a_n = 1 \ &\Rightarrow\ x_n > 1/2. \end{aligned}$$

By LPO, either $a_n = 0$ for all sufficiently large n or else $a_n = 1$ for infinitely many n. Thus either $x_n \in \left[0, \frac{1}{2}\right]$ for infinitely many n or else $x_n \in \left[\frac{1}{2}, 1\right]$ for infinitely many n. Repeating this argument, we can push through the classical interval–halving proof of the Bolzano–Weierstraß theorem. *q.e.d.*

We can now complete the **Proof of Proposition 3.3.1.** Assuming 3.1, we see from Lemmas 3.3.2 and 3.3.4 that the sequence $(t_n)_{n=1}^{\infty}$ contains a subsequence converging to a limit $t \in [0,1]$ such that

$$|t - t_z| \geq \delta. \tag{3.2}$$

Writing $y = t\xi + (1-t)\,z$, we see that $h\left(z_{n_k}\right) \to y$ as $k \to \infty$. Since ∂Y is closed and contains $h\left(z_{n_k}\right)$ for each k, it contains y; whence $y \neq \xi$ (as $\xi \in Y^{\circ}$) and therefore $t < 1$. On the other hand, by the definition of t_z, we have $t \geq t_z$; whence $t > t_z$, by (3.2). Thus

$$y = \theta\xi + (1-\theta)\,h(z),$$

where

$$0 < \theta = \frac{t - t_z}{1 - t_z} \leq 1.$$

Choosing $r > 0$ such that $B(\xi, r) \subset Y$, consider any $y' \in B(y, \theta r)$ and let

$$x = \frac{1}{\theta}\left(y' - (1-\theta)\,h\,(z)\right)$$

We have

$$\|x - \xi\| = \frac{1}{\theta}\,\|y' - y\| < r,$$

so $x \in Y^{\circ}$ and therefore, by the convexity of Y,

$$y' = \theta x + (1-\theta)h(z) \in Y.$$

Hence $B(y, \theta r) \subset Y$ and therefore $y \in Y^{\circ}$, which is absurd since $y \in \partial Y$. It follows that we must, in fact, have $t = t_z$, a contradiction. Thus (3.1) is ruled out, and our proof of Proposition 3.3.1 is finished. *q.e.d.*

In order to upgrade from sequential continuity to continuity for the mapping h, we have three more lemmas.

Lemma 3.3.5 *∂Y is located in $\mathbb{R}^N$.*

Proof. Since Y is located and convex, $-Y$ is located ([74], page 244, Proposition (1.5)). But Y is closed, so by Bishop's Lemma ([9], page 98, Lemma 3), $-Y = \sim Y$ The desired conclusion now follows from Proposition 11 of [40]. *q.e.d.*

Lemma 3.3.6 *Let $z \in -Y$, $\delta > 0$, and $\overline{B}(z, \delta) \subset -Y$. Let C be the compact cone joining ξ to $\overline{B}(z, \delta)$. Then*

$$C \cap \partial Y = \{h(x) : \|x - z\| \leq \delta\}.$$

Proof. For each $x \in \overline{B}(z,\delta)$, the convexity of C ensures that $h(x) \in C \cap \partial Y$. Conversely, given $y \in C \cap \partial Y$, extend the ray from ξ through y until it intersects $\overline{B}(z,\delta)$ in a point x; then $y = h(x)$. *q.e.d.*

The next lemma is needed because there is no guarantee that a general compact cone with vertex ξ has compact intersection with the boundary of Y.

Lemma 3.3.7 *Let $z \in -Y$, $\delta > 0$, and $\overline{B}\,(z,\delta) \subset -Y$. Then there exists $\delta' \in (0,\delta)$ such that $C' \cap \partial Y$ is compact, where C' is the cone joining ξ to $\overline{B}\,(z,\delta')$.*

Proof. Let C be the compact cone joining ξ to $B = \overline{B}(z,\delta)$. For each x in $-B\,(\xi, r/2)$, let $P(x)$ be the projection of x on the line segment $[\xi, z]$, and let

$$\theta(x) = \tan^{-1}\frac{\|x - P(x)\|}{\|P(x) - \xi\|}.$$

(Thus $\theta(x)$ is the radian measure of the angle between the ray joining ξ to x and the ray joining ξ to z.) The mapping θ is uniformly continuous on $-B(\xi, r/2)$ and therefore on $\partial Y - B(\xi, r/2)$. By Theorem (4.9) on page 98 of [9], there exists ε such that $0 < 2\varepsilon < \delta$ and

$$K = \{x \in \partial Y : \varepsilon \le \theta(x) \le \delta - \varepsilon\}$$

is compact. Using elementary Euclidean geometry, we now compute $\delta' \in (0,\delta)$ such that if C' is the cone joining ξ to $\overline{B}(z,\delta')$, then $K = C' \cap \partial Y$. *q.e.d.*

We now complete the **Proof of Theorem 3.1.1.** We need to prove the continuity of h. Accordingly, fix z in $\sim Y^\circ$. We first consider the case where $z \in -Y$. Using Lemmas 3.3.6 and 3.3.7, compute a strictly decreasing sequence $(\delta_n)_{n=1}^{\infty}$ of positive numbers with $\delta_1 < \rho\,(z, Y)$, such that

$$K_n = \{h(\zeta) : \|\zeta - z\| \le \delta_n\}$$

is compact for each n. Let

$$s_n = \sup \{\|h(\zeta) - h(z)\| \in K_n\}.$$

Given $\varepsilon > 0$, construct an increasing binary sequence $(\lambda_n)_{n=1}^{\infty}$ such that

$$\lambda_n = 0 \ \Rightarrow \ s_n > \varepsilon/2,$$
$$\lambda_n = 1 \ \Rightarrow \ s_n < \varepsilon.$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, choose ζ_n such that $\|\zeta_n - z\| \leq \delta_n$ and $\|h(\zeta_n) - h(z)\| > \varepsilon/2$. If $\lambda_n = 1 - \lambda_{n-1}$, set $\zeta_k = \zeta_{n-1}$ for all $k \geq n$. Then $(\zeta_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\overline{B}(z, \delta_1)$ and so converges to a point $\zeta_\infty \in \overline{B}(z, \delta_1)$. Either $h(\zeta_\infty) \neq h(z)$ or $\|h(\zeta_\infty) - h(z)\| < \varepsilon/2$. In the latter case we must have $\lambda_n = 0$ for all n, which contradicts Proposition 3.3.1. Hence $h(\zeta_\infty) \neq h(z)$ so $\zeta_\infty \neq z$, by Proposition 3.2.7, and therefore we can find N such that $\|\zeta_n - z\| > \delta_N$ for all $n \geq N$. It follows that $\lambda_N = 1$; whence $\|h(x) - h(z)\| < \varepsilon$ whenever $\|x - z\| < \delta_N$. This completes the proof of the continuity of h on $-Y$.

Now consider the general case, where $z \in \sim Y^\circ$. Using the convexity of Y, construct $z' \in -Y$ such that $z \in [\xi, z']$. Given $\varepsilon > 0$, and using the first part of the proof, we can compute δ' with $0 < \delta' < \rho(z', \partial Y)$, such that if $x \in \overline{B}(z', \delta')$, then $\|h(x) - h(z')\| < \varepsilon$. Let C be the compact cone joining ξ to $\overline{B}(z', \delta')$. Compute $\delta > 0$ such that $\overline{B}(z, \delta) \subset C$. If $x \in \sim Y^\circ$ and $\|x - z\| \leq \delta$, then there exists x' such that $\|x' - z'\| \leq \delta'$ and $h(x) = h(x')$; whence

$$\|h(x) - h(z)\| = \|h(x') - h(z')\| < \varepsilon$$

and our proof is finished. *q.e.d.*

Chapter 4

Foundations of Optimisation Theory

4.1 Introduction

As was first shown by Brouwer, although a uniformly continuous function on a compact—that is, complete, totally bounded—metric space has an infimum, it cannot be proved constructively that the infimum is attained (is a minimum). In fact, there is a recursive example of a uniformly continuous mapping f of $[0, 1]$ onto $(0, 1]$ ([38], Chapter 6).

It seems hard to produce interesting conditions that ensure the attainment of the infimum of a uniformly continuous map from a compact space into $\mathbb{R}$. However, as we show in the first part of this chapter, the proof of a theorem of Ekeland ([3], pages 15–16) can be adapted to yield conditions guaranteeing the existence of strong forms of approximate minima.

In the second part of the chapter we investigate conditions which are equivalent to the (local) total boundedness of the strict lower sections (which will be defined shortly) of a function. Altogether, this chapter can be viewed as a first step towards constructive optimisation theory, complementing related work on approximation theory such as that in [10].

4.2 Ekeland's Theorem

Before dealing with Ekeland's Theorem, we introduce some notations and definitions. For convenience, we use $\mathbf{F}$ to denote either the real line $\mathbb{R}$ or the extended real line $\overline{\mathbb{R}}$, and $\mathrm{dom}\,(f)$ to denote the domain of a function f.

Let f be a partial mapping of a metric space X into $\mathbb{R}$. For each $\lambda \in \mathbf{F}$ we define the corresponding

- **strict lower section**

$$S^{\mathrm{sl}}(f,\lambda) = \{x \in \mathrm{dom}\,(f) : \lambda < f(x)\}$$

  and

- **upper section**

$$S^{\mathrm{u}}\,(f,\lambda) = \{x \in \mathrm{dom}\,(f) : f(x) \leq \lambda\}$$

of f. The **strict upper section** $S^{\mathrm{su}}\,(f,\lambda)$ and the **lower section** $S^{\mathrm{l}}\,(f,\lambda)$ of f are defined analogously. Note that although we allow our function f to take only real values, it is convenient to allow the parameter λ to take the extended real value $-\infty$.

The constructive proof of Ekeland's theorem depends on an important theorem of Bishop ([9], page 98, Theorem (4.9)), which we state in an extended form that is easily proved as a consequence of Bishop's original version.

Theorem 4.2.1 *Let X be a compact metric space, and $f : X \to \mathbf{F}$ a uniformly continuous function. Then for all but countably many real numbers λ, the upper* *and lower sections $S^{\mathrm{u}}(f,\lambda)$ and $S^{\mathrm{l}}(f,\lambda)$ are compact, and the strict upper and strict lower sections $S^{\mathrm{su}}(f,\lambda)$ and $S^{\mathrm{sl}}(f,\lambda)$ are totally bounded.*

We are now ready for **Ekeland's Theorem:**

Theorem 4.2.2 *Let f be a uniformly continuous mapping of a compact metric space X into the nonnegative real line $\mathbb{R}^{0+}$, let $x_0 \in X$, and let $\varepsilon, \delta > 0$. Then there exists $\xi \in X$ such that*

$$f(\xi) + \varepsilon\rho(\xi, x_0) \leq f(x_0) \tag{4.1}$$

and

$$f(\xi) \leq f(x) + \varepsilon\rho(\xi, x) + \delta \tag{4.2}$$

for all $x \in X$.

Proof. For simplicity, we take $\varepsilon = 1$. We construct an increasing binary sequence $(\lambda_n)_{n=0}^{\infty}$, a sequence $(\varepsilon_n)_{n=0}^{\infty}$ of positive numbers, a sequence $(x_n)_{n=1}^{\infty}$ of elements of X, and a sequence $(F_n)_{n=0}^{\infty}$ of compact subsets of X such that the following properties hold for each $n \geq 0$:

(i) $0 < \varepsilon_n < \min\{\delta, 2^{-n}\}$.

(ii) If $\lambda_n = 0$, then

$$F_n = \{x \in X : f(x) + \rho(x, x_n) \leq f(x_n) - \varepsilon_n\},$$

$x_{n+1} \in F_n$, and

$$f(x_{n+1}) < \inf_{x \in F_n} f(x) + 2^{-n}. \tag{4.3}$$

(iii) If $\lambda_n = 1 - \lambda_{n-1}$, then for each $k \geq n$, $F_k = \{x_n\}$ and $x_k = x_n$.

Using Theorem 4.2.1, choose ε_0 such that $0 < \varepsilon_0 < \min\{\delta, 1\}$ and

$$K_0 = \{x \in X : f(x) + \rho(x, x_0) \leq f(x_0) - \varepsilon_0\}$$

is compact or empty. If K_0 is empty, set $\lambda_k = 1$, $F_k = \{x_0\}$, and $x_k = x_0$ for each k. If K_0 is compact, set $\lambda_0 = 0$ and $F_0 = K_0$. Having found λ_n, F_n, and ε_n, if $\lambda_n = 0$, then choose x_{n+1} in F_n such that (4.3) holds, and use Theorem 4.2.1 to obtain ε_{n+1} such that $0 < \varepsilon_{n+1} < \min\{\varepsilon_n, 2^{-n-1}\}$ and

$$K_{n+1} = \{x \in X : f(x) + \rho(x, x_{n+1}) \leq f(x_{n+1}) - \varepsilon_{n+1}\}$$

is compact or empty. If K_{n+1} is compact, set $\lambda_{n+1} = 0$ and $F_{n+1} = K_{n+1}$. If K_{n+1} is empty, for each $k \geq n+1$ set $\lambda_k = 1$, $F_k = \{x_{n+1}\}$, and $x_k = x_{n+1}$. This completes the inductive construction.

We may assume that $\lambda_0 = 0$. Write

$$v_n = \inf_{x \in F_n} f(x).$$

If $n \geq 1$ and $\lambda_n = 0$, then for each $x \in F_n$ we have

$$\begin{aligned} f(x) + \rho(x, x_{n-1}) &\leq f(x) + \rho(x, x_n) + \rho(x_n, x_{n-1}) \\ &\leq f(x_n) - \varepsilon_n + \rho(x_n, x_{n-1}) \\ &\leq f(x_{n-1}) - \varepsilon_{n-1} - \varepsilon_n \qquad (as\ x_n \in F_{n-1}) \\ &< f(x_{n-1}) - \varepsilon_{n-1}. \end{aligned}$$

Hence $F_n \subset F_{n-1}$. Moreover, in that case we have, for each $x \in F_n$,

$$v_n + \rho(x, x_n) \leq f(x) + \rho(x, x_n) \leq f(x_n) - \varepsilon_n$$

and so

$$\rho(x, x_n) \leq f(x_n) - v_n - \varepsilon_n.$$

Since F_n is compact, we see that $\operatorname{diam}(F_n)$ exists (as the supremum of the uniformly continuous function $d_i : F_n \times F_n \to \mathbb{R}$, defined by $d_i(x,y) = \rho(x,y)$) and is less than $2(f(x_n) - v_n)$. Since $F_n \subset F_{n-1}$,

$$0 \leq f(x_n) \leq v_{n-1} + 2^{-n+1} \leq v_n + 2^{-n+1};$$

whence

$$\operatorname{diam}(F_n) \leq 2(f(x_n) - v_n) \leq 2^{-n+2}.$$

On the other hand, if $\lambda_n = 1$, then for some $k \leq n$,

$$F_n = \cdots = F_{k+2} = F_{k+1} = \{x_k\} \subset F_k$$

and $\operatorname{diam}(F_n) = 0$.

We now see that $(F_n)_{n=0}^{\infty}$ is a descending sequence of compact sets whose diameters tend to 0; whence $\bigcap_{n=0}^{\infty} F_n$ consists of a single point $\xi \in X$. Moreover, $\rho(F_n, \{\xi\}) \to 0$, where ρ denotes the Hausdorff metric on the set of compact subsets of X. Since $\xi \in F_0$ and $\lambda_0 = 0$, statement (4.1) holds by the definition of F_0. Consider any $x \in X$, and suppose that

$$f(x) + \rho(x,\xi) < f(\xi) - \delta. \tag{4.4}$$

If there exists $n \geq 1$ such that $\lambda_n = 1 - \lambda_{n-1}$, then our construction of x_n and F_n shows that $\xi = x_n$ and

$$f(x) + \rho(x,\xi) = f(x) + \rho(x,x_n) \geq f(x_n) - \varepsilon_n \geq f(\xi) - \delta,$$

a contradiction. Hence $\lambda_n = 0$ for all n. Since f is strongly extensional, we see from (4.4) that $x \neq \xi$. It follows that for all sufficiently large n, $\rho(x, F_n) > \frac{1}{2}\rho(x,\xi)$ and therefore

$$f(x) + \rho(x,x_n) \geq f(x_n) - \varepsilon_n > f(x_n) - \delta.$$

Letting $n \to \infty$ and using the continuity of f, we obtain the contradiction $f(x) + \rho(x, \xi) \geq f(\xi) - \delta$. It now follows that this last inequality actually holds, as therefore does (4.2). *q.e.d.*

We now arrive at a consequence of Ekeland's theorem which provides us with rather strong approximate minima.

Corollary 4.2.3 *Let f be a uniformly continuous mapping of a compact metric space X into $\mathbb{R}^{0+}$, let $\varepsilon, \delta, r > 0$, and let x_0 be a point of X such that $f(x_0) < \inf_{x \in X} f(x) + \varepsilon r$. Then there exists $\xi \in X$ such that $f(\xi) \leq f(x_0)$, $\rho(\xi, x_0) < r$, and $f(\xi) \leq f(x) + \varepsilon \rho(\xi, x) + \delta$ for all $x \in X$.*

Proof. Applying Theorem 4.2.2, construct $\xi \in X$ such that

$$f(\xi) + \varepsilon \rho(\xi, x_0) \leq f(x_0)$$

and

$$f(\xi) \leq f(x) + \varepsilon \rho(\xi, x) + \delta \quad (x \in X).$$

Then $f(\xi) \leq f(x_0)$ and (since $\inf_{x \in X} f(x) \geq 0$)

$$\rho(\xi, x_0) \leq \frac{f(x_0)}{\varepsilon} < r.$$

q.e.d.

Consider the recursive example of a uniformly continuous mapping f from $[0,1]$ onto $(0,1]$, mentioned in the first paragraph of this chapter. Let $x_0 \in [0,1]$, $0 < \varepsilon < f(x_0)$, and $\delta > 0$. Since $\inf f = 0$, there exists $\xi \in [0,1]$ such that

$$f(\xi) < \min\{f(x_0) - \varepsilon, \delta\}.$$

We have

$$f(\xi) + \varepsilon\rho\,(\xi, x_0) \leq f(\xi) + \varepsilon < f(x_0)$$

and, for each $x \in [0, 1]$,

$$f(\xi) \leq \delta \leq f(x) + \varepsilon\rho\,(\xi, x) + \delta.$$

Thus for this function, given x_0 and δ, we can obtain the conclusion of Ekeland's Theorem *for all sufficiently small* ε, without using the theorem itself.

The classical version of Ekeland's theorem applies with "compact" and "uniformly continuous" replaced by "complete" and "lower semicontinuous", respectively. In both the intuitionistic and the recursive models of constructive mathematics, every (total) mapping of a complete metric space into a metric space is continuous—uniformly so in the first model, although not in the second. So lower semicontinuity is a generalisation of continuity on a complete space in only one of the three standard models of our constructive mathematics: the classical model. Nevertheless, it is worth noting that the last part of our proof of Theorem 4.2.2 does go through with continuity replaced by an appropriate notion of lower semicontinuity.

To see this, we first define a partial mapping $f : X \longrightarrow \mathbb{R}$ on a metric space to be **lower semicontinuous** if $\mathrm{L}\,(f)$ is open in $\mathrm{dom}\,(f) \times \overline{\mathbb{R}}$. In that case, $\mathrm{U}\,(f)$ is closed in X; moreover, f is strongly extensional (see Definition 3.2.6).

Proposition 4.2.4 *Let $f, g : X \longrightarrow \mathbb{R}$ be lower semicontinuous partial functions. Then $f + g$, cf (c a nonnegative constant) and* $\inf\{f, g\}$ *are lower semicontinuous.*

If $(f_i)_{i \in I}$ is a family of lower semicontinuous functions on X, then $\sup\limits_{i \in I} f_i$ *is lower semicontinuous.*

Proof. Let $\lambda_0 < f\,(x_0) + g\,(x_0)$, and choose $\varepsilon > 0$ such that $\lambda + \varepsilon < f(x_0) + g(x_0)$. Since $\mathrm{L}\,(f)$ is open and contains $(x_0, \lambda_0 + \varepsilon - g(x_0))$, there exists $\delta_1 > 0$ such that if $x \in \mathrm{dom}\,(f)$, $\lambda \in \overline{\mathbb{R}}$, $\rho\,(x, x_0) < \delta_1$, and $\rho\,(\lambda, \lambda_0) < \delta_1$, then $\lambda + \varepsilon - g(x_0) < f(x)$.

On the other hand, $(x_0, g(x_0) - \varepsilon) \in \mathrm{L}(g)$, so by our hypotheses, there exists $\delta_2 > 0$ such that if $x \in \mathrm{dom}(g)$, $\lambda \in \overline{\mathbb{R}}$, $\rho(x, x_0) < \delta_2$, and $\rho(\lambda, \lambda_0) < \delta_2$, then $g(x_0) - \varepsilon < g(x)$. Set $\delta = \min\{\delta_1, \delta_2\}$. If $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, $\lambda \in \overline{\mathbb{R}}$, $\rho(x, x_0) < \delta$, and $\rho(\lambda, \lambda_0) < \delta$, then

$$\lambda < f(x) + g(x_0) - \varepsilon < f(x) + g(x).$$

Hence $\mathrm{L}(f+g)$ is open.

The remaining parts of the proof are relatively straightforward and are omitted.

q.e.d.

Now let us look at the last part of the proof of Theorem 4.2.2 when f is assumed to be only lower semicontinuous. Assuming (4.4), we see that everything goes through as before until we reach the step where

$$f(x_n) - \rho(x, x_n) \leq f(x) + \delta$$

for all sufficiently large n. Since f is lower semicontinuous and $\rho(x, \cdot)$ is continuous, $f + \rho(x, \cdot)$ is lower semicontinuous, by Proposition 4.2.4. Thus $\mathrm{U}(f - \rho(x, \cdot))$ is closed in $\mathrm{dom}(f)$. Letting $n \to \infty$, we obtain $f(\xi) - \rho(x, \xi) \leq f(x) + \delta$ and therefore $f(x) + \rho(x, \xi) \geq f(\xi) - \delta$. This contradicts (4.4).

The only other place that we used the continuity of f in the proof of Theorem 4.2.2 was in the inductive construction of the sets F_n, to ensure that v_n existed when F_n was nonempty. So we see that the following extension of Theorem 4.2.2 holds constructively:

Corollary 4.2.5 *Let f be a continuous mapping of a complete metric space X into $\mathbb{R}^{0+}$, and suppose that there exist dense subsets D and E of* $\mathrm{dom}(f)$ *and* $\mathbb{R}$ *respectively, such that for all $\zeta \in D$ and $\varepsilon \in E$ the set*

$$F(\zeta, \varepsilon) = \{x \in X : f(x) + \rho(x, \zeta) \leq f(\zeta) - \varepsilon\}$$

has a computable infimum. Let $x_0 \in X$ and $\varepsilon, \delta > 0$. Then there exists $\xi \in X$ such that

$$f(\xi) + \varepsilon\rho(\xi, x_0) \leq f(x_0)$$

and

$$f(\xi) \leq f(x) + \varepsilon\rho(\xi, x) + \delta$$

for all $x \in X$.

4.3 Epigraphs, subgraphs and sections

Throughout this section, (X, ρ) is a metric space, and f a strongly extensional partial mapping of X into $\mathbb{R}$ with nonempty[1] domain. We define the **subgraph** $\mathrm{L}(f)$ and the **epigraph** $\mathrm{U}(f)$ as follows:

$$\begin{aligned}
\mathrm{L}(f) &= \{(x, \lambda) : x \in \mathrm{dom}(f), \lambda \in \mathbf{F}, \lambda < f(x)\}, \\
\mathrm{U}(f) &= \{(x, \lambda) : x \in \mathrm{dom}(f), \lambda \in \mathbf{F}, f(x) \leq \lambda\}.
\end{aligned}$$

Bearing in mind the importance of Theorem 4.2.1—for example, in the proof of Theorem 4.2.2—we investigate various connections between $\mathrm{L}(f)$, $\mathrm{U}(f)$, and the sections of f.

The comments about Markov's Principle in the introduction show that our next proposition is not the triviality that classical logic suggests.

Proposition 4.3.1 $\mathrm{L}(f) = \left(\mathrm{dom}(f) \times \overline{\mathbb{R}}\right) \sim \mathrm{U}(f)$.

Proof. Let $\lambda, \lambda' \in \overline{\mathbb{R}}$, $x, x' \in \mathrm{dom}(f)$, $\lambda < f(x)$, and $f(x') \leq \lambda'$. Then $f(x') - \lambda' \leq 0 < f(x) - \lambda$, so $f(x') - f(x) < \lambda' - \lambda$ and therefore either $f(x') - f(x) < 0$

[1] Recall that a set S is **nonempty**—in Brouwer's term, **inhabited**—if there exists an element that belongs to S. This is a stronger property than $\neg(S = \emptyset)$.

or $0 < \lambda' - \lambda$. In the first case, $f(x) \neq f(x')$ and so, as f is strongly extensional, $x \neq x'$. In the second case, $\lambda \neq \lambda'$. Thus in either case, $(x, \lambda) \neq (x', \lambda')$ in the product metric space $\mathrm{dom}\,(f) \times \overline{\mathbb{R}}$.

Conversely, if

$$(x, \lambda) \in \left(\mathrm{dom}\,(f) \times \overline{\mathbb{R}}\right) \sim \mathrm{U}\,(f)\,,$$

then, since $(x, f(x)) \in \mathrm{U}\,(f)$, we have $(x, \lambda) \neq (x, f(x))$; it follows that $\lambda \neq f(x)$ and therefore, since $\neg\,(f(x) \leq \lambda)$, that $\lambda < f(x)$. *q.e.d.*

Proposition 4.3.2 *If* $\mathrm{U}\,(f)$ *is totally bounded, then* $\inf\{f\,(x) : x \in \mathrm{dom}\,(f)\}$ *exists in* $\mathbb{R}$.

Proof. The second projection $p_2 : \mathrm{U}\,(f) \to \mathbb{R}$, $p_2(x, \lambda) = \lambda$ is a uniformly continuous mapping of the totally bounded set $\mathrm{U}\,(f)$ into $\mathbb{R}$, so

$$m = \inf\{\lambda \in \mathbb{R} : \exists x \in D\ (f(x) \leq \lambda)\}$$

exists in $\mathbb{R}$. Clearly, $f(x) \leq m$ for all $x \in D$. On the other hand, for each $\varepsilon > 0$ there exist $x \in D$ and $\lambda \in \mathbb{R}$ such that $f(x) \leq \lambda < m + \varepsilon$. Hence $m = \inf\{f(x) : x \in D\}$. *q.e.d.*

A locally totally bounded subspace of a metric space (see Definition 2.2.2) is located, and every located subspace of a locally totally bounded space (see Definition 2.2.2) is locally totally bounded ([38], page 33, (4.11)).

Proposition 4.3.3 *If* $\mathrm{L}\,(f)$ *is locally totally bounded, then* $\mathrm{dom}\,(f)$ *is locally totally bounded.*

Proof. Let B be a bounded subset of $D = \mathrm{dom}(f)$. Then $B \times \{-\infty\}$ is a bounded subset of $\mathrm{L}(f)$, so there exists a totally bounded subset S of $\mathrm{L}(f)$ such that $B \times \{-\infty\} \subset S$. Now, the first projection mapping $\phi : \mathrm{L}(f) \to X$, defined by $\phi(x, \lambda) = x$, maps $\mathrm{L}(f)$ onto D and $B \times \{-\infty\}$ onto B. Hence

$$B = \phi(B \times \{-\infty\}) \subset \phi(S) \subset D.$$

Since ϕ is uniformly continuous, $\phi(S)$ is totally bounded. Thus every bounded subset of D is contained in a totally bounded subset of D. *q.e.d.*

The following can be regarded as a general form of Theorem 4.2.1.

Proposition 4.3.4 *The following are equivalent conditions on a partial mapping f of X into $\mathbb{R}$:*

(i) *For all λ in a dense subset of $\mathbb{R}$, $S^{\mathrm{sl}}(f, \lambda)$ is either locally totally bounded or empty;*

(ii) *$L(f)$ is locally totally bounded;*

(iii) *For all but countably many $\lambda \in \mathbb{R}$, $S^{\mathrm{sl}}(f, \lambda)$ is either locally totally bounded or empty.*

Proof. Fix (x_0, λ_0) in $\mathrm{L}(f)$. Assuming (i), construct a countable dense subset C of $\mathbb{R}$ consisting of real numbers λ such that $S^{\mathrm{sl}}(f, \lambda)$ is either locally totally bounded or empty. For each $\lambda \in C$ such that $S^{\mathrm{sl}}(f, \lambda)$ is locally totally bounded, choose a sequence $(r_{\lambda,k})_{k=1}^{\infty}$ of positive numbers such that if $r > 0$ and $r \neq r_{\lambda,k}$ for each k, then $S^{\mathrm{sl}}(f, \lambda) \cap B(x_0, r)$ is totally bounded. On the other hand, if $\lambda \in C$ and $S^{\mathrm{sl}}(f, \lambda)$ is empty, set $r_{\lambda,k} = 1$ for each k. Since the numbers $r_{\lambda,k}$, with λ in C and $k \geq 1$, can be listed in a single sequence, we can find a strictly increasing sequence

$(r_n)_{n=1}^{\infty}$ of positive numbers tending to ∞ such that for each n,

$$r_n \neq r_{\lambda,k} \quad (\lambda \in C,\, k \geq 1)$$

and therefore $S^{\mathrm{sl}}(f,\lambda) \cap B(x_0, r_n)$ is either empty or totally bounded. It is enough to prove that for any positive integer ν the set

$$A = \mathrm{L}(f) \cap B((x_0, \lambda_0), r_\nu)$$

is totally bounded. To this end, given $\varepsilon > 0$, choose elements

$$\lambda_1 = -r_\nu < \lambda_2 < \cdots < \lambda_{n-1} < r_\nu = \lambda_n$$

of C such that $\lambda_{k+1} - \lambda_k < \varepsilon$ $(1 \leq k < n)$. If $S^{\mathrm{sl}}(f,\lambda_k)$ is locally totally bounded, choose a finite ε–approximation F_k to $S^{\mathrm{sl}}(f,\lambda_k) \cap B(x_0, r_\nu)$; if $S^{\mathrm{sl}}(f,\lambda_k)$ is empty, set $F_k = \emptyset$. Then

$$F = \{(x,\lambda) : \exists k \; (x \in F_k \wedge \lambda = \lambda_k)\}$$

is a finite subset of A. Given (ξ, λ) in A, choose k such that $1 \leq k < n$ and $\lambda_k \leq \lambda < \lambda_{k+2}$. Then $\xi \in S^{\mathrm{sl}}(f,\lambda_k) \cap B(x_0, r_\nu)$, so there exists $x \in F_k$ such that $\rho(x, \xi) < \varepsilon$. Also

$$0 < \lambda - \lambda_k < \lambda_{k+2} - \lambda_k < \varepsilon,$$

and so $\rho((\xi,\lambda),(x,\lambda_k)) < \varepsilon$. Since $(x, \lambda_k) \in F$, it follows that F is a finite ε–approximation to A. Hence A is totally bounded, and so (i) $\Rightarrow$ (ii).

Now assume (ii) and choose a strictly increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers diverging to ∞ such that

$$B_n = \mathrm{L}(f) \cap B((x_0, \lambda_0), r_n)$$

is totally bounded for each n. Since the projection maps pr_k $(k = 1, 2)$ are uniformly continuous on $\mathrm{L}(f)$, for each n there exists a sequence $(\lambda_{n,k})_{k=1}^{\infty}$ in $\overline{\mathbb{R}}$ such that if $\lambda \in \overline{\mathbb{R}}$ and $\lambda \neq \lambda_{n,k}$ for each k, then

$$S^{\mathrm{sl}}(f,\lambda,n) = \{x : (x,\lambda) \in B_n\} = \mathrm{pr}_1\left(\mathrm{pr}_2^{-1}(\lambda,\infty)\right)$$

is totally bounded ([9], page 98, Theorem (4.9), and page 94, Prop. (4.2)). Let $(\lambda_j)_{j=1}^{\infty}$ be an enumeration of all the extended real numbers $\lambda_{n,k}$ $(n, k \geq 1)$, and let λ be an extended real number such that $\lambda \neq \lambda_j$ for each j. Let B be a bounded subset of $S^{\mathrm{sl}}(f, \lambda)$, and choose N such that $B \subset B(x_0, r_N)$. Then B is contained in $S^{\mathrm{sl}}(f, \lambda, N)$, which, by our choice of λ, is a totally bounded subset of $S^{\mathrm{sl}}(f, \lambda)$. Thus (ii) $\Rightarrow$ (iii). Finally, it is trivial that (iii) $\Rightarrow$ (i). *q.e.d.*

Corollary 4.3.5 *If X is locally compact, and f a total mapping of X into $\mathbb{R}$ that is uniformly continuous on bounded sets, then* $\mathrm{L}(f)$ *is locally totally bounded.*

Proof. It is enough to prove that f satisfies condition (iii) of the preceding proposition. Fixing x_0 in X, choose a strictly increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers tending to ∞ such that $K_n = \overline{B}(x_0, r_n)$ is compact for each n. By Theorem (4.9) on page 98 of [9], for each n there exists a sequence $(\lambda_{n,k})_{k=1}^{\infty}$ of real numbers such that

$$S^{\mathrm{sl}}(f, \lambda) \cap K_n = \{x \in K_n : \lambda < f(x)\}$$

is totally bounded whenever $\lambda \in \mathbb{R}$ and $\lambda \neq \lambda_{n,k}$ for each k. Let $(\lambda_i)_{i=1}^{\infty}$ be an enumeration of the numbers $\lambda_{n,k}$ $(n, k \geq 1)$ in a single sequence. If $\lambda \in \mathbb{R}$ and $\lambda \neq \lambda_i$ for each i, then $S^{\mathrm{sl}}(f, \lambda)$ is the union of the totally bounded subsets

$$S^{\mathrm{sl}}(f, \lambda) \cap K_n \, (n \geq 1).$$

q.e.d.

The proofs of our final two results are similar to those of the preceding two and are omitted.

Proposition 4.3.6 *The following are equivalent conditions on a partial mapping f of X into $\mathbb{R}$:*

(i) *For all λ in a dense subset of $\mathbb{R}$, $S^{\mathrm{u}}(f,\lambda)$ is either locally totally bounded or empty;*

(ii) *$U(f)$ is locally totally bounded;*

(iii) *For all but countably many $\lambda \in \mathbb{R}$, $S^{\mathrm{u}}(f,\lambda)$ is either locally totally bounded or empty.*

Corollary 4.3.7 *If X is locally compact, and f a total mapping of X into $\mathbb{R}$ that is uniformly continuous on bounded sets, then $\mathrm{U}(f)$ is locally compact.*

Chapter 5

Constructing Approximate Equilibria

The third chapter, on boundary crossings, dealt with a major problem in the constructivisation of the proof by McKenzie [56] that, under economically and mathematically reasonable hypotheses, competitive equilibria exist. We begin this chapter by outlining Takayama's presentation of McKenzie's classical proof [84] (pages 265–274), as a prelude to discussing the constructive problems that remain to be overcome.

The set–up for the proof is as in Chapter 2, whose notations we adopt for the rest of this chapter. So, for example, we have consumer i $(1 \leq i \leq m)$ with **consumption set** $X_i \subset \mathbb{R}^N$, preference relation $\succ_i$ (we do not need utility functions here), and initial endowment $\bar{x}_i$; producer j $(1 \leq j \leq n)$ with **production set** $Y_j \subset \mathbb{R}^N$; **aggregate consumption set**

$$X = \sum_{i=1}^{m} X_i,$$

aggregate production set

$$Y = \sum_{j=1}^{n} Y_j,$$

and **total initial endowment**

$$\bar{x} = \sum_{l=1}^{m} \bar{x}_l \in X.$$

We also introduce some new notions. Under the price p, consumer i has the following **budget set**

$$\beta_i(p) = \{x \in X_i : p \cdot x \leq 0\}.$$

The **upper contour set of the economy** is

$$C(p) = \sum_{i=1}^{m} C_i(p).$$

In view of the definition of $\beta_i(p)$, it makes sense to introduce into the discussion the following two sets:

- the **polar cone** of Y,

$$Y^* = \left\{p \in \mathbb{R}^N : \forall y \in Y \, (p \cdot y \leq 0)\right\}$$

and

- the **normalised polar** of Y,

$$P = \{p \in Y^* : p \cdot \bar{x} = -1\}$$

Consumer i's **demand at the price** p is defined by $f_i : P \to X_i$,

$$f_i(p) = C_i(p) \cap \beta_i(p),$$

and the **aggregate demand function** f is defined on $\mathbb{R}^N$ by

$$f(p) = \sum_{i=1}^{m} f_i(p).$$

We say that consumer i is

- **satiated with the bundle** x_i if $x_i \succcurlyeq_i x$ for all $x \in X_i$;

- **satiated with the bundle** x_i **at the price** $p \in P$ if $x_i \succcurlyeq_i x$ for all $x \in C_i(p)$.

The preference ordering $\succ_i$ is said to be **strictly convex 2** if X_i is convex and $x_i \succcurlyeq_i x_i$ and $x'_i \neq x_i$ imply that $tx'_i + (1-t)x_i \succ_i x_i$, for each $t \in (0,1)$.
Takayama's exposition of McKenzie's proof is based on the following six axioms (which we have adapted according to our definitions of such terms as "strictly convex 2"):

A1 Each consumption set X_i is convex, closed, and bounded.

A2 Each X_i is totally quasi–ordered by a strictly convex 2 and continuous preference ordering $\succ_i$.

A3 The set Y is a closed convex cone.

A4 $Y \cap \Omega = \emptyset$.

A5 For each i the set $X_i \cap Y$ has nonempty interior.

A6 For each $p \in P$, either no consumer is satiated at p, or else $C(p) \cap Y = \emptyset$.

We comment briefly on some of these axioms. For example, in axiom **A2**, which reflects the absence of technological external economies and diseconomies (interactions among production processes), the convexity of Y does not mean that each Y_j is convex. Axioms **A3** ("impossibility of the Land of Cockaigne") and **A4** together are intended to reflect that the total profit in the economy is zero—that is, if someone wins, someone else loses. Axiom **A5** guarantees that every consumer can supply a positive amount of every non–produced commodity to the producers. Axiom **A6** says that if some consumer is satiated while trading at price p, then the total demand at p will exceed the possible production.

Axioms **A1**, **A2**, and **A5** together ensure classically that $f_i(p)$ is a singleton subset of X_i and hence can be identified with its sole element; and that the resulting (point–point) demand function f_i is continuous on P.

Using a highly nonconstructive argument by contradiction that involves the sequential compactness of the unit ball of $\mathbb{R}^N$, Takayama proves that P is compact. He then introduces the boundary crossing map that we discussed in Chapter 3, the mapping $g : \partial Y \to 2^P$ defined by

$$g(z) = \{p \in P : p \cdot z = 0\},$$

and the mapping $F : P \to 2^P$ by

$$F(p) = (g \circ h \circ f)(p) \ \ (p \in P).$$

He shows by another nonconstructive argument that for each $z \in \partial Y$ the set $g(z)$ is compact—so that F maps each element of the compact set P to a compact subset of P—and that F is upper semicontinuous (as a set–valued mapping) in the classical sense. This enables him to apply the classical Kakutani fixed–point theorem to F, to produce a point $p \in P$ such that $p \in F(p)$. Unwrapping this last property, he shows that p is the required equilibrium price vector.

We have at least the following constructive problems with this proof:

- the proof that P is compact;
- the proof that for each $z \in \partial Y$ the set $g(z)$ is compact;
- the classical definition of "upper semicontinuous", which is difficult to apply in constructive mathematics;
- the lack of a constructive version of Kakutani's fixed–point theorem.

In connection with the last of these, it is worth noting that the Brouwer's fixed–point theorem, which is classically equivalent to Kakutani's (see [55], Appendix IV), does not hold constructively: indeed, in the one–dimensional case Brouwer's theorem is equivalent to the intermediate value theorem, so the best we can do in general is to construct approximate fixed points (see pages 8 and 40 of [9]). This strongly suggests that we should be aiming for a constructive proof of the existence of *approximate equilibria* rather than exact ones.[1]

With the preceding paragraph in mind, for each $\varepsilon \geq 0$ we introduce the ε–**approximate polar**

$$Y_\varepsilon^* = \{p \in \mathbb{R}^N : \forall y \in Y \, (p \cdot y \leq \varepsilon)\}$$

and the ε–**approximate normalised polar**

$$P_\varepsilon = \left\{p \in \mathbb{R}^N : p \cdot \bar{x} = -1 \wedge \sup_{y \in Y} p \cdot y \leq \varepsilon\right\} = Y_\varepsilon^* \cap P$$

of Y. Note that both Y_ε^* and P_ε are convex sets.

We aim to prove that P_ε is compact for all but countably many small $\varepsilon > 0$.

Lemma 5.0.8 *The set*

$$\{p \in \mathbb{R}^N : p \cdot \bar{x} = -1\}$$

is locally compact.

Proof. The bounded linear functional $u : \mathbb{R}^N \to \mathbb{R}$ defined on the Euclidean (Hilbert) space $\mathbb{R}^N$ by

$$u(p) = p \cdot \bar{x}$$

is normable, with norm $\|\bar{x}\|$. Since $\bar{x}$ is nonzero, $\ker u$ is located in $\mathbb{R}^N$, by [9] (Prop. (1.10), page 303). We construct $p_1 \in \mathbb{R}^N$ such that $p_1 \cdot \bar{x} = -1$. To do so, we first use [9] (Cor. (4.5), page 341) to construct a normable linear functional v on $\mathbb{R}^N$ such that $\|v\| = 1$ and $v(\bar{x}) > 0$. By the Riesz Representation Theorem ([9], Prop. (2.3),

[1]It is not correct to suggest, as has frequently been done, that the work of Scarf [80] enables one to compute exact equilibria: the most it can do is to produce approximate equilibria, as approximate fixed points of a certain set–valued mapping.

page 419), there exists $p_0 \in \mathbb{R}^N$ such that $v(x) = p_0 \cdot x$ for all $x \in \mathbb{R}^N$. Writing $p_1 = -p_0/v(x_0)$, we obtain

$$p_1 \cdot \bar{x} = \frac{-1}{v(x_0)} p_0 \cdot \bar{x} = -1.$$

Since

$$p \cdot \bar{x} = -1 \Leftrightarrow p - p_1 \in \ker u,$$

it follows that

$$u^{-1}(-1) = \left\{p \in \mathbb{R}^N : p \cdot \bar{x} = -1\right\} = p_1 + \ker u.$$

Hence

$$\rho\left(p, u^{-1}(-1)\right) = \rho\left(p - p_1, \ker u\right) \quad \left(p \in \mathbb{R}^N\right),$$

so $u^{-1}(-1)$ is located in $\mathbb{R}^N$ and therefore locally compact. *q.e.d.*

Proposition 5.0.9 *P_ε is locally compact for all but countably many $\varepsilon > 0$.*

Proof. Since the mapping $\phi : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\phi(p) = \sup_{y \in Y} p \cdot y$$

is uniformly continuous on the locally compact subset $u^{-1}(1)$ of $\mathbb{R}^N$, we see from [9] (Theorem (4.9), page 98), extended to the locally compact case, that for each positive integer n there exists a sequence $(\varepsilon_{n,k})_{k=1}^{\infty}$ of positive numbers such that the set

$$P_\varepsilon \cap \overline{B}(0,n) = \left\{p \in u^{-1}(1) : \sup_{y \in Y} p \cdot y \leq \varepsilon\right\}$$

is compact whenever $\varepsilon > 0$ and $\varepsilon \neq \varepsilon_{n,k}$ for each k. Thus if $\varepsilon > 0$ and $\varepsilon \neq \varepsilon_{n,k}$ for all n and k, the set $P_\varepsilon \cap \overline{B}(0,n)$ is compact for each positive integer n. The result follows. *q.e.d.*

Lemma 5.0.10 *Let N, r be positive numbers, and let $0 < \varepsilon < \frac{r}{\sqrt{N}}$. Then*

$$\frac{2\varepsilon}{r\sqrt{N}} - \frac{\varepsilon^2}{r^2} > 0 \tag{5.1}$$

and if

$$0 < \alpha < \frac{2\varepsilon}{r\sqrt{N}} - \frac{\varepsilon^2}{r^2}, \tag{5.2}$$

then

$$\sqrt{\frac{1}{N} - \alpha} > \frac{1}{\sqrt{N}} - \frac{\varepsilon}{r}.$$

Proof. Since

$$\frac{\varepsilon}{r} - \frac{2}{\sqrt{N}} < 0,$$

we easily obtain (5.1). Now let (5.2) hold. Then

$$\left(\frac{1}{\sqrt{N}} - \frac{\varepsilon}{r}\right)^2 = \frac{1}{N} - \frac{2\varepsilon}{r\sqrt{N}} + \frac{\varepsilon^2}{r^2} < \frac{1}{N} - \alpha,$$

from which the desired conclusion follows. *q.e.d.*

Lemma 5.0.11 *Let $r \in (0,1)$ be such that $\overline{B}(\bar{x}, r) \subset Y$, let $0 < \varepsilon < \frac{r}{3\sqrt{N}}$ be such that P_ε is locally compact, and let p be a unit vector in Y_ε^*. Then $|p \cdot \bar{x}| \geq \frac{r}{\sqrt{N}}$.*

Proof. Noting that $\frac{r}{3\sqrt{N}} < \frac{2r}{\sqrt{N}}$, we see from Lemma 5.0.10 that we can find α such that (5.2) holds. Since

$$\sum_{i=1}^{N} p_i^2 = 1 > N\left(\frac{1}{N} - \alpha\right),$$

there exists i such that $p_i^2 > \frac{1}{N} - \alpha$. Taking, for example, the case

$$p_i > \sqrt{\frac{1}{N} - \alpha},$$

we see from Lemma 5.0.10 that

$$p_i > \frac{1}{\sqrt{N}} - \frac{\varepsilon}{r}.$$

For $1 \leq j \leq N$ define

$$y_j = \begin{cases} \bar{x}_j & j \neq i \\ \\ \bar{x}_i + r & j = i. \end{cases}$$

Then $y = (y_1, \ldots, y_N) \in Y$. Suppose that $|p \cdot \bar{x}| < \frac{r}{\sqrt{N}}$. Then

$$\varepsilon \geq p \cdot y = p \cdot \bar{x} + p_i r > -\varepsilon + \left(\frac{r}{\sqrt{N}} - \frac{\varepsilon}{r}\right) = \frac{r}{\sqrt{N}} - 2\varepsilon > \varepsilon,$$

which is absurd. Hence $|p \cdot \bar{x}| \geq \frac{r}{\sqrt{N}}$. *q.e.d.*

Proposition 5.0.12 *Let $r > 0$ be such that $\overline{B}(\bar{x}, r) \subset Y$, and let ε be a positive number such that $0 < \varepsilon < \frac{r}{3\sqrt{N}}$ and P_ε is locally compact. Then P_ε is bounded and hence compact.*

Proof. Fix p_0 in P_ε. Construct a strictly increasing sequence $(r_n)_{n=0}^{\infty}$ of positive integers converging to ∞, such that $r_1 > 1$ and each of the sets $B_n = P_\varepsilon \cap \overline{B}(p_0, r_n)$ is compact. For each $n \geq 1$ let

$$s_n = \sup \{\rho(p, B_{n-1}) : p \in B_n\},$$

which exists by the uniform continuity of the function $\rho_1 : B_n \to \mathbb{R}^+$ given by

$$\rho_1(p) = \rho(p, B_{n-1})$$

on the compact set B_n. Choose ν such that

$$\frac{1 + \varepsilon}{r_\nu} < \frac{r}{\sqrt{N}}.$$

Supposing that $s_{\nu+1} > 0$, choose $p_\nu \in B_{\nu+1}$ such that $\|p_\nu\| > r_\nu$. Then $\bar{p} = \frac{p_\nu}{\|p_\nu\|} \in Y_\varepsilon^*$, since $p_\nu \in P_\varepsilon \subset Y_\varepsilon^*$ and Y_ε^* is closed under multiplication by numbers in $[0,1]$. Also,

$$|\bar{p} \cdot \bar{x}| = \frac{|p_\nu \cdot \bar{x}|}{\|p_\nu\|} \leq \frac{1+\varepsilon}{r_\nu} < \frac{r}{\sqrt{N}}.$$

Since this contradicts Lemma 5.0.11, we conclude that $s_{\nu+1} = 0$. If there exists $p \in P_\varepsilon$ with $\|p - p_0\| > r_\nu$, then, by the convexity of P_ε, there exists $p' \in P_\varepsilon$ such that $r_\nu < \|p - p_0\| < r_{\nu+1}$, so $s_{\nu+1} > 0$. This contradiction ensures that $P_\varepsilon \subset B_\nu$ and hence that P_ε, being both bounded and locally compact, is compact. *q.e.d.*

Thus for all sufficiently small positive ε—call them **admissible**—the set P_ε is compact. We now have another classical proof that P is compact: choosing admissible numbers $\varepsilon_n > 0$ such that

$$\frac{r}{3\sqrt{N}} > \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n \to 0,$$

we see that P, being the intersection of the decreasing sequence of compact sets P_{ε_n}, is itself compact. This, however, is not a constructive proof; to make it constructive, we would need to show that the sequence of diameters of the sets P_{ε_n} converges in $\mathbb{R}$.

It is time we looked more closely at the constructive counterpart of Takayama's axioms **A1**–**A6**. To do so, we refer the reader to the definition of "strictly convex" in Chapter 2 and introduce one more notion.

A convex subset X of a normed space V is said to be **uniformly rotund** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if x and y belong to X, $\|x - y\| \geq \varepsilon$, and z is an element of V with $\|z\| \leq \delta$, then $\frac{1}{2}(x+y) + z \in X$.

In order to apply the existence theorem for demand functions found in [17], we replace Takayama's first two axioms by the following:

BP1 For each i, the set X_i is compact and uniformly rotund in $\mathbb{R}^N$.

BP2 For each i the preference relation $\succ_i$ is continuous and strictly convex 2.

His next three axioms are constructivised as follows:

BP3 The aggregate consumption set Y is a locally compact, convex cone with vertex 0 in $\mathbb{R}^N$.

BP4 For each nonzero $y = (y_1, \ldots, y_N)$ in Y there exists k such that $y_k < 0$.

BP5 For each i there exists $\bar{x}^i \in (X_i \cap Y)^\circ$.

We then write

$$\bar{x} = \bar{x}^1 + \cdots + \bar{x}^m.$$

Since we work with the approximate normalised polar P_ε, it makes sense to ε**–approximate budget sets** of the form

$$\beta_i(p, \varepsilon) = \{x \in X_i : p \cdot x \leq \varepsilon\},$$

rather than the sets $\beta_i(p)$ used by Takayama. We also define ε**–approximate upper contour sets**

$$C_i(p, \varepsilon) = \{x_i \in X_i : \forall x \in \beta_i(p, \varepsilon)\ (x_i \not\prec_i x)\}$$

and

$$C(p, \varepsilon) = \sum_{i=1}^{m} C_i(p, \varepsilon).$$

There are at least two ways in which we can constructivise Takayama's axiom A6: namely,

BP6a If some consumer is satiated with some bundle at the price $p \in P_\varepsilon$, then $C(p, \varepsilon) \cap Y = \emptyset$

and

BP6b If $p \in P_\varepsilon$ and $C(p,\varepsilon) \cap Y \neq \emptyset$, then there exist i and a bundle $x_i \in X_i$ such that consumer i is satiated with x_i at p.

It is not clear which of these is preferable.

In order to apply the main theorem in [17], we need an extra hypothesis:

BP7 For each sufficiently small $\varepsilon > 0$ and each $p \in P_\varepsilon$, there exists $\xi_i \in X_i$ such that $\xi_i \succ_i x$ for all $x \in \beta_i(p,\varepsilon)$.

We seek an ε–**approximate equilibrium 2**: that is, a triple (ξ, η, p) of vectors in $\mathbb{R}^N$ such that

▷ for each i, $\xi_i \in C_i(p,\varepsilon) \cap \beta_i(p,\varepsilon)$;

▷ $p \in Y_\varepsilon^*$, $\eta \in Y$, and $p \cdot \eta = 0$;

▷ $\sum_{i=1}^{m} \xi_i = \eta$.

If $\varepsilon > 0$ is admissible, and we apply the main theorem in [17] to the sets $\beta_i(p,\varepsilon)$ where $p \in P_\varepsilon$ (note that we need **BP7** to do this), then for each $p \in P_\varepsilon$ and each i, we obtain a unique element $f_i(p,\varepsilon) \in \beta_i(p,\varepsilon)$ such that $f_i(p,\varepsilon) \succcurlyeq_i x$ for all $x \in \beta_i(p,\varepsilon)$; moreover, $p \cdot f_i(p,\varepsilon) = \varepsilon$, and the mapping $f_i : P_\varepsilon \to X_i$—called the ith **demand function**—defined by

$$f_i(p,\varepsilon) = C_i(p,\varepsilon) \cap \beta_i(p,\varepsilon),$$

is uniformly continuous on P_ε.

We recall the boundary crossing map $h :\sim Y^\circ \to \partial Y$, defined by

$$h(z) = t_z \xi + (1 - t_z)\, z.$$

We have already shown in Chapter 3 that h is well defined and pointwise continuous on $\sim Y^\circ$. In parallel with Takayama's development, for each admissible $\varepsilon > 0$ we introduce the set–valued function $g : \partial Y \to 2^P$ by setting

$$\begin{aligned} g_\varepsilon(z) &= \{p \in P_\varepsilon : p \cdot z = 0\} \\ &= P_\varepsilon \cap \{z\}^\perp \\ &= \left\{p \in \{z\}^\perp : \sup_{y \in Y} p \cdot y \leq \varepsilon \wedge p \cdot \bar{x} = -1\right\}. \end{aligned}$$

Lemma 5.0.13 *If* $z \neq 0$, *then*

$$S = \left\{p \in \{z\}^\perp : p \cdot \bar{x} = -1\right\}$$

is locally compact.

Proof. Since $z \neq 0$, we see that $\mathbb{R}z$, and therefore $\{z\}^\perp$, is located in $\mathbb{R}^N$. It follows that the linear functional $u : \mathbb{R}^N \to \mathbb{R}$ given by

$$u(p) = p\bar{x},$$

when restricted to $\{z\}^\perp$, is normable, its norm being that of the projection of $\bar{x}$ on $\{z\}^\perp$. The kernel $\{z\}^\perp \cap \ker u$ of this restricted functional is therefore located in $\{z\}^\perp$; whence, as in the proof of Lemma 5.0.8,

$$S = \left\{p \in \{z\}^\perp : p \cdot \bar{x} = -1\right\}$$

is located in $\{z\}^\perp$. Since $z \neq 0$, the set $\{z\}^\perp$ is located in $\mathbb{R}^N$ and therefore locally compact. In turn, it follows that S, being located in $\{z\}^\perp$, is locally compact. *q.e.d.*

Proposition 5.0.14 *If* $z \neq 0$, *then the set* $P_\varepsilon \cap \{z\}^\perp$ *is locally compact for all but countably many* $\varepsilon > 0$.

Proof. With S as in the preceding lemma, we have

$$P_\varepsilon \cap \{z\}^\perp = \left\{p \in S : \sup_{y\in Y} p \cdot y \leq \varepsilon\right\}.$$

Also, the mapping ϕ is uniformly continuous on S, which is locally compact by Lemma 5.0.13. By an extension to locally compact spaces ([9], page 98, Theorem (4.9)), for each positive integer n, there exists a sequence $(\varepsilon_{n,k})_{k=1}^{\infty}$ of positive numbers such that the set

$$P_\varepsilon \cap \{z\}^\perp \cap \overline{B}(0,n) = \left\{p \in S : \|p\| \leq n, \ \sup_{y\in Y} p \cdot y \leq \varepsilon\right\}$$

is compact for each positive ε such that $\varepsilon \neq \varepsilon_{n,k}$ for all n and k. Thus if $\varepsilon > 0$ and $\varepsilon \neq \varepsilon_{n,k}$ for all n and k, then $P_\varepsilon \cap \{z\}^\perp \cap \overline{B}(0,n)$ is compact for each positive integer n. *q.e.d.*

Corollary 5.0.15 *Let $r > 0$ be such that $\overline{B}(\bar{x}, r) \subset Y$, and let ε be a positive number such that $0 < \varepsilon < \frac{r}{3\sqrt{N}}$ and $P_\varepsilon \cap \{z\}^\perp$ is locally compact. Then $P_\varepsilon \cap \{z\}^\perp$ is compact.*

Proof. By Proposition 5.0.14, $P_\varepsilon \cap \{z\}^\perp$ is locally compact for all but countably many positive ε. By Proposition 5.0.12, P_ε is bounded for all sufficiently small ε. Since G_ε is a subset of P_ε, it follows that for all but countably many sufficiently small ε, $P_\varepsilon \cap \{z\}^\perp$ is both locally compact and bounded, and therefore compact.

q.e.d.

The problem with the preceding results is that the admissible values of ε depend on z. We really want to show that for all but countably many ε with $0 < \varepsilon < \frac{r}{3\sqrt{N}}$, and for all $z \in \mathbb{R}^N$, the set $P_\varepsilon \cap \{z\}^\perp$ is compact. The required argument is elusive.

It is tempting to conjecture that if, like P_ε for each admissible ε, C is a compact convex subset of $\mathbb{R}^N$, then $C \cap \{z\}^\perp$ is compact for all $z \neq 0$. Here, is a Brouwerian counterexample to that conjecture. Take $N = 2$, let θ be very close (and possibly equal) to 0, let $z = (\cos\theta, \sin\theta)$, and let

$$C = \{(0, y) : 0 \leq y \leq 1\}.$$

Then $z \neq 0$, and C is compact and convex, and intersects $\{z\}^{\perp}$ at 0. Suppose that $C \cap \{z\}^{\perp}$ is compact. Then

$$d = \sup\left\{\|x\| : x \in C \cap \{z\}^{\perp}\right\}$$

exists. Either $d > 0$ or $d < 1$. In the first case, $\neg(\theta = 0)$; whereas in the second, $\theta = 0$.

So if we are to prove that $P_{\varepsilon} \cap \{z\}^{\perp}$ is compact for all sufficiently small positive ε, independently of z, it seems that we will have to use some information about P_{ε} beyond its compactness and convexity.

Let us assume for the moment that we have established that $P_{\varepsilon} \cap \{z\}^{\perp}$ is compact, independently of z, for all sufficiently small admissible $\varepsilon > 0$. We then define $g_{\varepsilon} : \partial Y \to 2^{P_{\varepsilon}}$ by

$$g_{\varepsilon}(z) = P_{\varepsilon} \cap \{z\}^{\perp} \quad (z \in \partial Y,\ \varepsilon > 0),$$

and $f^{\varepsilon} : P_{\varepsilon} \to \sim Y^{\circ}$ by

$$f^{\varepsilon}(p) = \sum_{i=1}^{m} f_i(p, \varepsilon).$$

Finally, we define $F_{\varepsilon} : P_{\varepsilon} \to 2^{P_{\varepsilon}}$ by

$$F_{\varepsilon}(p) = (g_{\varepsilon} \circ h \circ f^{\varepsilon})(p).$$

In order to ensure that f^{ε} is well defined, at least for small enough admissible ε, we need to prove that if $p \in P_{\varepsilon}$, then $f^{\varepsilon}(p) \in\sim Y^{\circ}$. This is easy: the proof of Lemma 5.0.11 shows that if $p \in P_{\varepsilon}$ and $y \in Y^{\circ}$, then $p \cdot y < 0$; but $p \cdot f_i(p, \varepsilon) = \varepsilon$, so $p \cdot f^{\varepsilon}(p) = N\varepsilon > 0$ and therefore $p \neq y$.

We would then hope to apply a Kakutani–type (approximate) fixed–point theorem to construct an approximate equilibrium 2 for our economy. In order to apply such a theorem, we may need to have the boundary crossing map h uniformly continuous on compact sets, since Brouwer's fixed–point theorem, a special case of Kakutani's, definitely requires uniform continuity in the constructive setting (there are recursive counterexamples with only pointwise continuity; see Chapter 4 of [4]).

We conclude that, though the foregoing results and proofs about ε–approximate polars may be interesting—and, for the proof of existence of approximate equilibria 2, valuable—there remain major constructive problems en route to a full proof of the existence of those equilibria.

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